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BINDING ENERGY OF NUCLEAR MATTER

by

CHINN-CHANN CHIANG

A THESIS

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The undersigned certify that they have read,
and recommend to the Faculty of Graduate Studies for
acceptance, a thesis entitled BINDING ENERGY OF NUCLEAR
MATTER, submitted by Chinn-Chann Chiang, in partial
fulfilment of the requirements for the degree of
Master of Science.

ABSTRACT

The effective interaction in nuclear matter is defined as the long range part of the two body potential, which in the Born approximation gives the single particle potential energy for the average momentum in the Fermi sea. For the Gammel - Thaler potential the effective interaction has been calculated, first for the free particle propagator and then for the nuclear spectrum. The result shows that in the free particle case the separation distance ξ_0 is constant over a wide range of densities and does not lead to saturation. The nuclear separation distance $\xi(K_f)$ changes quite rapidly with the Fermi momentum K_f , and while for low densities it is very close to ξ_0 , for higher densities it becomes very much larger. At the density corresponding to $K_f = 1.5F^{-1}$, the long range potential starts at $\xi = 1.166F^{-1}$. The minimum of the total energy per particle occurs at $K_f = 1.43F^{-1}$ and is about -10MeV.

A variational method is also used to calculate the reaction amplitude for nuclear matter. It is found that the potential energy for 1S_0 state is -15.45 MeV as compared to -14. MeV obtained by perturbation.

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CONTENTS

| | Page |
|---|------|
| Chapter 1 Introduction | 1 |
| Chapter 2 The First Order Approximation | 12 |
| Chapter 3 First and Second Order Correlations | 26 |
| Chapter 4 Numerical Calculation with BGT Potential | 34 |
| A. Details of Calculations and Results | 34 |
| B. Discussion | 54 |
| Chapter 5 Variational Method | 58 |
| Chapter 6 Conclusion | 65 |

LIST OF TABLES

| | Page |
|--|------|
| I Parameters of the BGT Potential | 34 |
| II Phase Shifts Calculated from the Phase Functions | 35 |
| III Phase Shifts Calculated by Gammel and Thaler from 4205 Potential | 36 |
| IV The Separation Distances for 3S_1 State | 46 |
| V The Contribution of Different Waves to the Average Energy per Particle. | 47 |
| VI The Values of Effective Mass, $\gamma^2(k)$ and the Constants for Energy Denominator. | 48 |
| VII The First Order Correction of Separation Distances, $\xi(F)^{-1}$, and Contribution Potential Energies (MeV). | 51 |
| VIII The Contribution of the Second Order Terms to the Potential Energy. | 52 |
| IX The total Energy per Particle (MeV) Versus Fermi Momentum. | 53 |

LIST OF FIGURES

| Figure 1. | Page |
|---|------|
| a. The Phase Function of 1S_0 State as a function of $r(F)$ for $K_f = 1.5F^{-1}$. | 37 |
| b. The Phase Function of 3D_1 State as a Function of $r(F)$ for $K_f = 1.5F^{-1}$. | 38 |
| c. The Phase Function of 3D_2 State as a Function of $r(F)$ for $K_f = 1.5 F^{-1}$. | 39 |
| d. The Phase Functions of 1P_1 and 3P_1 State as a Function of $r(F)$ for $K_f = 1.5F^{-1}$. | 40 |
| e. The Phase Functions of 3P_0 and 3P_2 States as a Function of $r(F)$ for $K_f = 1.5F^{-1}$ | 41 |
| Figure 2. | |
| The Single Particle Energy as a Function of Relative Momentum for $K_f = 1.5F^{-1}$. | 49 |
| Figure 3. | |
| The Kinetic Energy, Potential Energy (in- cluding first and second order corrections) and Binding Energy per Particle as a Function of K_f^2 . | 56 |
| Figure 4. | |
| The Reaction Amplitude J as a Function of Trial Function Parameter, x , for 1S_0 state. | 64 |

Chapter I.

INTRODUCTION

The study of the properties of nuclei in its most general terms includes a large variety of topics ranging from the study of nuclei in their ground states to scattering and reaction processes at high energies. In this paper we shall concern ourselves with a much more restricted aspect of nuclei, the binding energy of nuclear matter.

The semi-empirical mass formula developed by Bethe and Weizsäcker around 1935-36 describes the "averaged binding energy curve of atomic nuclei with atomic mass A and atomic number Z ."

$$\text{Binding Energy} = -a_1 A + a_2 A^{2/3} + a_3 Z^2 A^{-1} + 1/4 a_4 (A/2 - Z)^2 A^{-1}$$

$$\simeq - (8\text{MeV}) A$$

where a_1 is a measure of the true nuclear interaction energy, a_2 measures the "unsaturated" nuclear binding energy at the nuclear surface, a_3 is determined by the

Coulomb energy between protons and a_4 is a repulsive term which says that the most stable nuclei have $Z \approx A/2$. Green⁽¹⁾ has made the most accurate fit to the semi-empirical mass formula. In particular, he has found the value $a_1 = 15.5$ MeV.

The concept of nuclear matter is an idealization based on the notion that a large number of nucleons would form a stable configuration if Coulomb repulsion between protons did not exist. The properties of this configuration will be independent of the number of nucleons if that number is so large that surface effects can be neglected. From the semi-empirical formula it is easy to see that the binding energy per nucleon is equal to $-a_1$ because for stable nuclei with large number of nucleons all of the terms except the first term can be neglected.

Nuclear matter may be looked upon as a gas of strongly interacting Fermi particles, and an individual particle or shell model of heavy nuclei consists of a description in terms of momentum states with an altered energy - momentum relation. Ordinary perturbation theory, with free particles as the unperturbed system, serves as a crude starting point by which one might attempt to establish a theoretical basis for the shell model. In such a treatment, we would take the entire interaction

as a perturbation. This was done by Euler as early as 1937, and later by Huby. The results were discouraging because the calculated binding energy was too small and the convergence rate was poor.

Three more successful independent calculations of the binding energy of nuclear matter have been made with realistic nuclear forces, by Brueckner and Masterson⁽²⁾, by Razavy⁽³⁾ following the reference spectrum method developed by Bethe, Brandow and Petschek⁽⁴⁾ and by Scott and Moszkowski⁽⁵⁾.

The potential energy of the individual particles, at least sufficient for finding the total nuclear energy, might be determined by a Hartree-Fock self-consistent field method. However, the evident inadequacy of ordinary perturbation theory and the nature of the nuclear forces (hard cores, exchange character, etc.) leads one to believe that in any accurate calculation one must include some of the strong correlations between the particles. Brueckner and co-workers⁽⁸⁾ have developed a method of treating the many body problem which takes a first step in this direction.

The theory of Brueckner and co-workers permits, in principle, calculation of properties of complex nuclei in terms of the potential between two nucleons. The theoretical foundation of the theory has been given by

Goldstone⁽⁶⁾ and by Hugenholtz⁽⁷⁾. Goldstone uses an expansion of the development operator whereas Hugenholtz expands the resolvent operator. Brueckner and Gammel⁽⁹⁾, aside from further developing the method did extensive numerical calculations, using the best results then available in the interaction between two nucleons, and obtained results in good agreement with experiment. Brueckner and Masterson⁽²⁾ used an approximation of the Brueckner formalism which is simpler than that of Brueckner - Goldstone.

The Brueckner-Goldstone (BG) method is motivated by the fact that the nuclear forces turn strongly repulsive at short distances. This makes any kind of perturbation treatment inapplicable, since the matrix elements of an infinite repulsive core potential with respect to independent particle wave functions are all infinite. However, it is well known that a potential containing a hard core does not have an infinite effect on the wave function of the interacting pair. In particular, we obtain a finite phase shift. The two-particle wave function is simply "pushed away" from the region of the core.

Brueckner's theory assumes that while the correlations between any two particles can be quite large, any correlations involving more than two particles at

a time can be neglected. This seems to be a quite good approximation for nuclear matter. Accordingly, the nuclear potential energy is just the sum of the interaction energies of each pair of nucleons and the latter is determined in terms of the two-particle scattering properties.

The Brueckner's theory can be criticized from two points of view: From the basic point of view, it may be questioned whether it actually leads to ground state of nuclear matter, and from the practical point of view the numerical calculations required are complicated and not very transparent. On the problem of the basic side, Luttinger, Kohn and Ward^(10,11) have shown that the BG method is indeed correct to all orders of perturbation theory if (1) the particles have spin $1/2$, (2) the interaction between them is isotropic, and (3) the Fermi surface without interaction is isotropic in momentum space. These conditions are evidently satisfied for nucleons interacting with central forces. For the case of nucleons interacting with tensor forces, they showed that the BG theory is correct in the second order. Another problem on the basic side is whether perturbation theory is valid. The chief argument of it is that nuclear matter may exhibit a phenomenon similar to superconductivity. Bethe⁽⁴⁾ has pointed out that the "superconductivity pairing" has a negligible influence on

the average binding energy of nuclei, so that the BG method can be considered as sufficiently established on theoretical grounds. However, the numerical work is complicated, especially due to the repulsive core in the nucleon interaction. Levinger, Peierls, Razavy and collaborators⁽¹³⁾ have therefore suggested to ignore the repulsive core and to replace its effect by a velocity dependent potential. There is no objection, in principle, against the use of velocity-dependent potentials. Actually a priori it is hard to tell whether a velocity-dependent or a static potential gives better results. But Charap, Fubini, and Tausner⁽¹⁴⁾ have shown that at least at moderate energies, dispersion theory leads to an unambiguous definition of a static potential which can be calculated in terms of exchange of one, two, etc., pions (see however Ref. 15). In nuclear matter, the filled states have momenta up to $k_F = 1.5 F^{-1}$ which is moderate. So that an ordinary, static, nucleon-nucleon potential should give a good description of nuclear matter.

A great simplification over the direct solution of Brueckner integral equation was made by Moszkowski and Scott⁽¹⁵⁾ (MS). They developed an approximation scheme which makes use of some of the physical features of the nuclear forces. The essential thing is that the nuclear forces are basically attractive at moderately low energies. Although they apparently turn strongly repulsive at short distances, the attraction at larger

distances more than compensates this repulsion. In other words, the repulsion effectively cancels a part, but not all, of the attraction.

The nucleon-nucleon interaction is therefore separated into a short-range part, V^S and a long-range part V^L . The short-range part is defined such that it gives zero phase shift for free particle scattering and is represented by a reaction matrix G_F^S which is not very different for nuclear matter and for free nucleons. The long-range part V^L is the remaining part of the interaction which can be treated by Born approximation in nuclear matter while for free nucleons due to the Pauli effect this is not possible. The separation between short and long range is made in such a way that the short range reaction matrix for free nucleons, G_F^S , vanishes. Then the total reaction matrix G consists mainly the first and second Born approximation for V^L , the difference between G^S in nuclear matter, G_N^S , and for free nucleons, G_F^S ; in addition there are some small corrections.

This method is much simpler than the direct solution of the reaction matrix of Brueckner-Goldstone, but Köhler⁽¹⁷⁾ finds that the MS calculation of $G_N^S - G_F^S$ may not be sufficiently accurate. (G_N^S is the nuclear reaction matrix.)

The MS method has been improved by Bethe et al (B.B.P.)⁽⁴⁾. They also have developed a method, the

reference spectrum method, which is even simpler than the MS method. They replace the single particle energy for the intermediate states by reference energy defined as $E_R(k') = A + k'^2/2m^*$ where m^* is an effective mass and k' is the single particle momentum. With this replacement the energy denominator will be a simple quadratic function of the relative momentum of the excited states. In addition, if the Pauli operator is also dropped the integral equation can be reduced to a second order differential equation which can be handled quite easily. The reference spectrum parameters, A and m^* , are chosen to fit the actual particle energies with the reference spectrum as accurate as possible for $k' > k_f$. The reference reaction matrix G^R is then used to calculate the Brueckner matrix G . The difference $G - G^R$ is due mainly to the effect of the exclusion. The spectral correction, which arises from the approximation of the energy denominator is quite small and shows that a detailed knowledge of the form of the single particle energy is not important for the calculation of the reaction matrix. The actual energy spectrum must of course be calculated from the reaction matrix.

The MS separation is useful to assess the sensitivity of the nuclear binding energy to the behaviour of the nucleon interaction at short distances. In the MS method,

the short range forces are mainly important for the determination of the separation distance d . We can also determine d , for a given relative momentum k_0 , from the observed phase shift of nucleon-nucleon scattering for momentum k_0 , if we assume that the long-range nuclear potential (say for distances greater than $1F$) is known. We can simply integrate the Schrödinger equation inwards from infinity (or some large value of r), until we reach the point at which the logarithmic derivative of the wave function is equal to that of free nucleons; this is then the MS separation distance for non-interacting nucleons, d_F .

Puff and Martin⁽¹⁸⁾ and Mohling⁽¹⁹⁾ have proposed a new method. In their method in the scattering into intermediate states the exclusion principle is not taken into account and the excited state energies are replaced by $k^2/2M$. The energies of holes are determined self-consistently as in other theories. The Puff-Martin theory involves partial summations and cannot be developed in perturbation theory, even when well behaved potentials are employed. This is because it leads from zero-order continuum states to a resulting state which is bound. It is well known that this cannot occur in perturbation theory taken to any finite order. On the other hand, the Brueckner theory can be expanded in perturbation theory when weak well-behaved potentials are employed.

The main task of this paper is to develop an easier method to calculate the Brueckner reaction matrix which minimizes the effect of the purely phenomenological part of the potential. As in the MS method, we separate the two body interaction into a short-range and a long-range part. But we modify the definition of separation point and treat the Hamiltonian of the short-range interaction as a perturbation. The separation distance is determined by the requirement that the first order contribution of the short-range potential to the reaction matrix be zero. This definition of separation distance differs from that of MS. With our separation the first order expression for the reaction matrix is just the first Born approximation for the long-range part of the potential and the energy of the system can be obtained from the effective Hamiltonian which contains the long-range interaction and the kinetic energy. The details of solving the first order Brueckner equation are contained in Chapter 2. The first order correction is mainly due to the reaction matrix of the short-range part treated as perturbation which is discussed in Chapter 3. In the same chapter the second order correction is also discussed. Chapter 4 contains the numerical calculation and comparison of the results with other works.

We also use a variational principle for calculating the reaction matrix based on Schwinger's method. Hulthen⁽²⁰⁾ has proposed a workable variational method which has been modified by Köhn⁽²¹⁾. Schwinger⁽²²⁾ approached the problem in a different way. While both Hulthen and Köhn used the differential form of the Schrödinger equation, Schwinger started out by taking the appropriate integral equation corresponding to the Schrödinger equation. This gave rise to a complicated functional involving Green's function.

In order to apply the Schwinger's method in nuclear matter problem we use the integral equation for the reaction matrix but with Green's function different from that of Schwinger's to calculate the reaction amplitude. This variational method is discussed in Chapter 5 and the result is compared with that of perturbation.

Chapter 2.

THE FIRST ORDER APPROXIMATION

The basic assumption in this theory is that the correlations between any two particles can be very large, while correlations involving more than two particles at a time are small and to the first approximation can be neglected. Thus the total Hamiltonian of the nuclear matter contains the sum of kinetic energy of the particles and two-body interactions and can be written as:

$$H = \sum_i T_i + 1/2 \sum_{i,j=1}^A V(\vec{r}_{ij}) \quad (2.1)$$

In equation (2.1) A is the number of nucleons, T_i the kinetic energy operator for the particle in the state i and $V(\vec{r}_{ij})$ the potential between two particles in state i and j respectively. The interaction can be divided into two parts, i.e. the long-range attractive part and the short-range part which is strongly repulsive for very small r and then strongly attractive. The separation distance which is not yet specified is denoted by ξ_{ij} . The Hamiltonian can be written as:

$$H = H_0 + H'$$

with

$$H_0 = \sum_i T_i + 1/2 \sum_{i,j} V(\vec{r}_{ij}) \theta(r_{ij} - \xi_{ij})$$

$$H' = 1/2 \sum_{i,j} V(\vec{r}_{ij}) \theta(\xi_{ij} - r_{ij}) \quad (2.2)$$

H_0 is unperturbed Hamiltonian and H' perturbation.

The total Hamiltonian H defines the exact wave vector $|\psi_0\rangle$ and the total energy E of the ground state of the system

$$H |\psi_0\rangle = E |\psi_0\rangle \quad (2.3)$$

and H_0 defines a set of unperturbed wave vectors $|\phi_n\rangle$ with eigenvalue E_n

$$H_0 |\phi_n\rangle = E_n |\phi_n\rangle. \quad (2.4)$$

We normalize $|\psi_0\rangle$ such that $\langle \phi_0 | \psi_0 \rangle = 1$. We can expand $|\psi_0\rangle$ in terms of the complete orthonormal set $|\phi_n\rangle$ to get

$$|\psi_0\rangle = |\phi_0\rangle + \sum_{n \neq 0} \frac{|\phi_n\rangle \langle \phi_n | H' | \psi_0 \rangle}{E - E_n} \quad (2.6)$$

and the total energy

$$E = E_0 + \langle \phi_0 | H' | \psi_0 \rangle$$

$$= E_0 + \Delta E. \quad (2.7)$$

The energy shift ΔE is a function of the separation distance ξ_{ij} . If $\xi_{ij} = c$ where c is the radius of the hard

core (region of strong repulsion), then $\Delta E > 0$. On the other hand if ξ_{ij} is very large then for the existence of the bound state ΔE should be negative. Thus it is possible to find a point ξ_{ij} in such a way that the energy shift vanishes.

The wave vector $|\psi_{ij}\rangle$ for each pair of particles is defined such that $|\psi_{ij}\rangle$ contains correlations of all pairs except ij

$$|\psi_{ij}\rangle = |\phi_0\rangle + \sum_{n \neq 0} \frac{|\phi_n\rangle \langle \phi_n| H' - V_{ij}^S |\psi_0\rangle}{E - E_n} \quad (2.8)$$

where $V_{ij}^S = V(\vec{r}_{ij}) \theta(\xi_{ij} - r_{ij})$

Define the operator G_{ij}^S by

$$G_{ij}^S |\psi_{ij}\rangle = V_{ij} |\psi_0\rangle \quad (2.9)$$

then from (2.6), (2.8) and (2.9) it follows that

$$|\psi_0\rangle = |\phi_0\rangle + \sum_{n \neq 0} \frac{|\phi_n\rangle \langle \phi_n| 1/2 \sum_{i,j} G_{ij}^S |\phi_0\rangle}{E - E_n} + \dots \quad (2.10)$$

Then with the approximation $|\psi_{ij}\rangle \cong |\phi_0\rangle$, we have the integral equation for G_{ij}^S

$$\langle \phi_0 | G_{ij}^S | \phi_0 \rangle = \langle \phi_0 | V_{ij}^S | \phi_0 \rangle + \sum_{n \neq 0} \frac{\langle \phi_0 | V_{ij}^S | \phi_n \rangle \langle \phi_n | G_{ij}^S | \phi_0 \rangle}{E - E_n}$$

If now the separation distance is specified by imposing the condition

$$\langle \Phi_0 | G_{ij}^S | \Phi_0 \rangle = 0, \quad (2.12)$$

then with the approximation $|\Psi_{ij}\rangle \cong |\Phi_0\rangle$, the energy shift is

$$\begin{aligned} \langle \Phi_0 | H' | \Psi_0 \rangle &= \frac{1}{2} \sum_{ij} \langle \Phi_0 | V_{ij}^S | \Psi_0 \rangle \\ &= \frac{1}{2} \sum_{ij} \langle \Phi_0 | G_{ij}^S | \Psi_{ij} \rangle \\ &= \frac{1}{2} \sum_{ij} \langle \Phi_0 | G_{ij}^S | \Phi_0 \rangle \\ &= 0, \end{aligned}$$

and the total energy of the system is

$$\begin{aligned} E = E_0 &= \sum_i \langle \Phi_0 | T_i | \Phi_0 \rangle + 1/2 \sum_{j \neq i} V_{ij}^\ell | \Phi_0 \rangle \\ &= \sum_i E_i - 1/2 \sum_{ij} \langle \Phi_0 | V_{ij}^\ell | \Phi_0 \rangle \end{aligned}$$

where $V_{ij}^\ell = V(\vec{r}_{ij}) \theta(r_{ij} - \xi_{ij})$ is the long-range part of the interaction potential and E_i is the single-particle energy

$$E_i = \langle \Phi_0 | T_i | \Phi_0 \rangle + \sum_{i \neq j} \langle \Phi_0 | V_{ij}^\ell | \Phi_0 \rangle$$

Equation (2.10) can be compared with the complete reaction matrix which in the notation used by Moszkowski, Scott has the following form

$$G = V + V \frac{Q}{e} G$$

$$\begin{aligned}
\text{with } Q(p, K') &= 0 & \text{If } K^2 + p^2/4 < K_f^2 \\
&= 1 & \text{If } K' - p/2 > K_f \\
&= \frac{K'^2 + p^2/4 - K_f^2}{K'p} & \text{otherwise}
\end{aligned}$$

The matrix element of G is

$$\langle K | G | K_0 \rangle = \langle K | V | K_0 \rangle + \langle K | V | K' \rangle \frac{Q}{e(K', K)} \langle K' | G | K_0 \rangle \quad (2.13)$$

The matrix elements of G are also given by

$$\langle \vec{K} | G | \vec{K}_0, \vec{p} \rangle = \langle \vec{K} | V | \psi^n(\vec{K}_0, \vec{p}) \rangle \quad (2.14)$$

Here \vec{K}_0 , \vec{K}' , \vec{K} denote the initial, intermediate, and final relative momentum of the two interacting nucleons, and \vec{p} is their average momentum so that the laboratory momenta of the two nucleons in the intermediate state are $\vec{p} + \vec{K}'$ and $\vec{p} - \vec{K}'$. $|\psi^n(\vec{K}_0, \vec{p})\rangle$ is the wave function for two nucleons interacting through the potential $V(\vec{r})$.

For brevity, the parameter \vec{p} will be omitted hereafter. From the definition of G^S , it is seen that equation (2.12) may be written as

$$\langle \vec{K}_0 | V^S | \psi(\vec{K}_0) \rangle = 0.$$

Thus the diagonal elements of G are given by

$$\langle \vec{k}_0 | G | \vec{k}_0 \rangle = \langle \vec{k}_0 | V^\ell | \psi^n(\vec{k}_0) \rangle$$

with approximation $|\psi\rangle = |\psi^n\rangle$ being made. It is well known(16) that at large distances (i.e. $r > \xi$) the free particle wave function is a good approximation to the two particle wave function in nuclear matter.

Thus to first order we have

$$\langle \vec{k}_0 | G | \vec{k}_0 \rangle \approx \langle \vec{k}_0 | V^\ell | \vec{k}_0 \rangle \quad (2.15)$$

To obtain the correct integral form of the reaction matrix we have to take into account the spin and isospin states of the two nucleons, which are required for the complete specification of the dynamical state of the two-particle system. In doing so, the reaction matrix can be written as

$$\langle \vec{k} | G | \vec{k} \rangle = \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \int_{\xi}^{\infty} j_{\ell}^2(Kr) V_{\ell,J}^{S,T}(r) r^2 dr \quad (2.16)$$

where $V_{\ell,J}^{S,T}$ is the potential depending on the total angular momentum J , orbital angular momentum ℓ , spin S and isospin T of the interacting nucleons.

The total energy of the system is then

$$\begin{aligned} E &= \sum_i T_i + 1/2 \sum_{i,j} \langle \vec{k} | G | \vec{k} \rangle \\ &= \sum_i \frac{k_i^2}{2} + 1/2 \sum_{i,j} \langle \vec{k} | G | \vec{k} \rangle \\ &= \sum_i [E(\vec{k}_i) - 1/2 U(\vec{k}_i)] \end{aligned} \quad (2.17)$$

where

$$\vec{K} = 1/2 (\vec{K}_i - \vec{K}_j)$$

$$U(\vec{K}_i) = \sum_{K_j < K_f} \langle \vec{K} | G | \vec{K} \rangle$$

The reaction matrix $\langle \vec{K} | G | \vec{K} \rangle$ is defined in terms of the relative momentum \vec{K} of the particles i and j . To find the average potential felt by the particles with momentum K_i , we calculate the average value of G when K_i is kept fixed and K_j takes on all the values in the Fermi sea. Then we multiply the result by the average density, $\rho = \frac{2K_f^3}{3\pi^2}$

to get

$$U(K_i) = \rho \int_{K_j < K_f} \langle K | G | K \rangle d^3K_j / \int_{K_j < K_f} d^3K_j \quad (2.18)$$

The average potential energy of the particle in the ground state, \bar{U} , can be written in terms of $U(K_i)$ as

$$\bar{U} = \int_{K_i < K_f} U(K_i) d^3K_i / \int_{K_i < K_f} d^3K_i \quad (2.19)$$

or directly as an integral of G

$$\bar{U} = \rho \int_0^{K_f} dK P(K, K_f) \langle K | G | K \rangle \quad (2.20)$$

where

$$P(K, K_f) = \frac{24 K^2}{K_f^3} \left(1 - \frac{3K}{2K_f} + \frac{K^3}{2K_f^3} \right)$$

is the normalized probability distribution of relative momentum for two particles in the Fermi sea. For low densities, instead of equation (2.20) for \bar{U} , we can use the reaction matrix evaluated at the average relative momentum \bar{K} multiplied by the density. The average relative momentum is a function of K_f which depends on the form of $\langle K | G | K \rangle$. Usually one uses $\bar{K} = \sqrt{0.3} K_f$ which is exact if $\langle K | G | K \rangle$ is a simple quadratic in K . Then \bar{U} becomes

$$\bar{U} = \frac{2}{3\pi^2} K_f^3 \langle \bar{K} | G | \bar{K} \rangle \quad (2.21)$$

The average kinetic energy \bar{T} of the particle in the Fermi sea is

$$\bar{T} = (3/10) K_f^2. \quad (2.22)$$

Thus the binding energy W per nucleon in nuclear matter in the first order is given by

$$W(K_f) = 3/10 K_f^2 + \frac{1}{3\pi^2} K_f^3 \langle \bar{K} | G | \bar{K} \rangle \quad (2.23)$$

or in integral form

$$W(K_f) = 3/10 K_f^2 + \frac{K_f^3}{6\pi} \sum_{J,T} (2J+1)(2T+1) \int_{\xi}^{\infty} j_{\ell}^2(\bar{K}r) V_{\ell,J}^{S,T}(r) r^2 dr \quad (2.24)$$

The integrations of the last part of (2.24) are easy to carry out for $\ell \leq 2$. For $\ell > 2$ the contribution to the reaction matrix is very small⁽³⁾ so we believe that it is good approximation to neglect the contribution for $\ell > 2$.

The solution of the reaction matrix is now reduced to the problem of calculating the separation distance ξ in a self-consistent way. A first order calculation can be based on the knowledge of the two-nucleon scattering phase shifts and the long-range part of the potential. The first order approximation, ξ_0 , is obtained by letting the interaction energy denominator tend to zero. Also the centre of mass of the two interacting particles is assumed to be at rest and the exclusion operator, Q , is replaced by the principle value operator. Then G_F^S , the first order approximation to G^S satisfies the equation

$$G_F^S = V^S + V^S \frac{P}{e_F} G_F^S$$

where $e^F(K_0, K') = K_0^2 - K'^2$ is the first approximation of e^N and P stand for Cauchy's principle value. This is just the equation for the reaction matrix with the potential

V^S and the solution is given in terms of the scattering phase shifts due to the short-range potential by

$$\langle K | G_F^S | K \rangle = - \frac{4\pi}{K} \sum_{\ell} (2\ell + 1) C_{\ell} \delta_{\ell}(K, \xi_0) \quad (2.27)$$

where $C_{\ell} = 3/4$ for even ℓ and $5/4$ for odd ℓ .

From the theory of scattering it is expected that the limit of the reaction matrix G_{ℓ}^S as the $e^N \rightarrow e^F$ tends to the t-matrix of scattering theory (for potential V^S) which is given in terms of $\tan \delta_{\ell}$ rather than δ_{ℓ} . However, Fukuda and Newton⁽²³⁾ have shown that as energy denominator tends to zero, the limit of G^S depends on δ_{ℓ} as shown in (2.27).

In the case of spin- and isospin-dependent potentials the G_F^S matrix is

$$\langle K | G_F^S | K \rangle = - \frac{\pi}{2K} \sum (2J + 1)(2T + 1) \delta_{\ell,J}^{S,T} \quad (2.28)$$

where $\delta_{\ell,J}^{S,T}$ is the phase shift due to the short-range part of the potential $V_{\ell,J}^{S,T}(r) \theta(\xi_0 - r)$. We require the first order separation distance, $\xi_0(K)$, be specified by the condition that the diagonal elements of G_F^S vanish

$$\langle K | G_F^S | K \rangle = 0. \quad (2.29)$$

To find the separation distance we define the phase function $\delta_{\ell,J}^{S,T}(K, r)$ by the differential equation⁽²⁴⁾

$$\frac{d}{dr} \delta_{\ell,J}^{S,T}(K,r) = -K r^2 V_{\ell,J}^{S,T} [\cos \delta_{\ell,J}^{S,T}(K,r) j_{\ell}(Kr) - \sin \delta_{\ell,J}^{S,T}(K,r) n_{\ell}(Kr)]^2 \quad (2.30)$$

with the boundary condition

$$\delta_{\ell,J}^{S,T}(K,\infty) = \delta_{\ell,J}^{S,T}(K) \quad (2.31)$$

where $\delta_{\ell,J}^{S,T}(K)$ are the known scattering phase shifts produced by the complete potential $V_{\ell,J}^{S,T}(r)$.

For practical purposes one prefers to integrate $\delta_{\ell,J}^{S,T}(K,r)$ from the origin to the point ξ_0 which is defined by eq. (2.29). In the case of a hard core potential of radius c , one starts at the core radius with the condition that the wave function should be zero inside the core. This implies that the phase function at the edge of the core is given by

$$\tan \delta_{\ell,J}^{S,T}(K,c) = j_{\ell}(Kc)/n_{\ell}(Kc) \quad (2.32)$$

One can also integrate the first few partial phase function (e.g. $\ell=0,1$ and 2) with no phase scattering from infinity to point ξ_0 where

$$\sum_{J,T} (2J+1)(2T+1) \delta_{\ell,J}^{S,T} = 0 \quad (2.33)$$

The solution $\xi_0(K)$ can be used to compute approximate values for the diagonal elements of the reaction matrix and the single-particle potential energy from eq. (2.16) and

(2.18). At first sight it may seem that the result is nearly the same as calculating G_F , i.e.

$$\langle K | G_F | K \rangle = - \frac{\pi}{2K} \sum_{J,T} (2J+1)(2T+1) \delta_{\ell,J}^{S,T} \quad (2.34)$$

But this is not the case and the difference between G calculated from (2.16) and (2.28) may be large. For example, the one-pion exchange (OPEP) contributes nothing to the G defined by (2.16), but it does contribute to G_F in (2.28). It should be noted that the approximate reaction matrix given by eq. (2.16) is a much better approximation for the actual nuclear matrix than G_F . The reason is that in nuclear matter the attractive potential cannot change the Fermi gas nearly so much as it affects free particles. Moszkowski and Scott⁽⁵⁾ have shown that G_F is a useful approximation only for large K .

To find the separation distance by eq. (2.33) one encounters difficulty for the coupled states due to the coupling by tensor force. For weak coupled states the tensor force can be neglected but for the strong coupled states such as ${}^3S_1 - {}^3D_1$, we can only find the separation distance by solving the Schrödinger equations.

The Schrödinger equation for ${}^3S_1 - {}^3D_1$ states are

$$\frac{d^2 u}{dr^2} + (K^2 - V_{11}(r))u = V_{12}w \quad (2.35)$$

$$\frac{d^2 w}{dr^2} + \left(K^2 - \frac{6}{r^2} - V_{22}(r) \right) w = V_{12}u \quad (2.36)$$

where u is the wave function of 3S_1 state and w of 3D_1 state. V_{12} is the tensor force, and V_{11} is the central potential. For 3D_1 , V_{22} is a combination of central, V_c , spin-orbit, V_{LS} , and tensor potential, V_T . The explicit forms of these potential are:

$$V_{11} = V_c$$

$$V_{12} = 2\sqrt{2} V_T$$

$$V_{22} = V_c - 2V_T - 3V_{LS}$$

The Schrödinger equation of free particle in S state is

$$\frac{d^2 \phi}{dr^2} + K^2 \phi = 0 \quad (2.37)$$

where ϕ is related to the spherical Bessel function as $\phi = rj_0(Kr)$. Equation (2.37) can be written as:

$$\frac{d^2 \phi}{dr^2} + (K^2 - V_{11}(r)) \phi = -V_{11}(r) \phi \quad (2.38)$$

Subtracting (2.35) from (2.38)

$$\frac{d^2}{dr^2} (\phi - u) + (K^2 - V_{11}(r)) (\phi - u) = (-V_{12}w - V_{11}\phi) \quad (2.39)$$

which can be written as

$$\frac{d^2}{dr^2} \chi + (K^2 - V_{11}) \chi = (-V_{12}w - V_{11}\phi) \quad (2.40)$$

where $\chi = \phi - u$ is the difference between the free particle wave function and the perturbed wave function.

If we put $w = -\zeta$ where w is the perturbed wave function of 3D_1 state, then we have the coupled differential equation:

$$\frac{d^2 \chi}{dr^2} + (K^2 - V_{11}) \chi = (V_{12} \zeta - V_{11} \phi) \quad (2.41)$$

$$\frac{d^2 \zeta}{dr^2} + (K^2 - \frac{6}{r^2} - V_{22}) \zeta = -V_{12} (\phi - \chi) \quad (2.42)$$

with the boundary condition

$$\chi(c) = \phi(c) \quad (2.43)$$

$$\zeta(c) = 0 \quad (2.44)$$

The first order separation distance ξ_0 is a point where the perturbed wave function joins smoothly to the unperturbed wave function. Since χ and χ' are continuous functions of r , it follows that

$$\chi(\xi_0) = 0 \quad (2.4)$$

and

$$\chi'(\xi_0) = 0 \quad (2.46)$$

The numerical results are given in Chapter 4.

Chapter 3.

First and Second Order Correlations

Once the approximate separation distance ξ_0 is obtained from (2.33), we can determine single-particle energies which enable us to find a more accurate value of the separation distance, ξ , by solving (2.10), subject to the condition that $\langle \Psi | G^S | \Psi \rangle = 0$. This can be done most easily by using the coordinate representation of the wave function. Assuming that $E - E_n$ in (2.7) depends only on the excitation energy of the two particles, we can reduce the equation to one describing the perturbed wave function of the two interacting particles:

$$| \Psi(Kr) \rangle = | \phi(Kr) \rangle + \sum_{K' > K_f} \frac{| \phi(K'r') \rangle \langle \phi(K'r') | V^S(r') | \Psi(Kr') \rangle}{e(K', K, p)} \quad (3.1)$$

Since the long-range part of the potential is rather weak, and also because of the exclusion principle, the distortion of the eigenfunctions of H_0 , $| \phi_0 \rangle = e^{i\vec{K} \cdot \vec{r}}$, are small. To a good approximation they can be regarded as plane waves. Thus

$$\Psi(\vec{K} \cdot \vec{r}) = \exp(i\vec{K} \cdot \vec{r}) + \frac{1}{(2\pi)^3} \int d^3 r' \int_{K' > K_f} d^3 K' \frac{\exp i\vec{K}' \cdot (\vec{r} - \vec{r}') V^S(\vec{r}') \Psi(\vec{K} \cdot \vec{r})}{e(K', K, p)} \quad (3.2)$$

which is similar to the Bethe-Goldstone equation.

To satisfy the condition $\langle \psi | G^S | \psi \rangle = 0$, the solution of (3.2) must satisfy the condition

$$\int \exp(-i\vec{K} \cdot \vec{r}) V^S \psi(\vec{K} \cdot \vec{r}) d^3r = 0 \quad (3.3)$$

The energy denominator in (3.2) is given by

$$\begin{aligned} e(K, K', p) &= E(\vec{K}_1, \vec{K}_j) - E(\vec{K}_m, \vec{K}_n) \\ &= E\left(\frac{1}{2} \vec{p} + \vec{K}, \frac{1}{2} \vec{p} - \vec{K}\right) - E\left(\frac{1}{2} \vec{p} + \vec{K}', \frac{1}{2} \vec{p} - \vec{K}'\right) \end{aligned}$$

where $E(K_1, K_j)$ and $E(K_m, K_n)$ are the eigenvalues of the two-particle Hamiltonian H_0 defined by relation similar to (2.1) and (2.2).

From (3.2) and (3.3) a better approximation for ξ can be found which in turn will determine $e(K', K, p)$ more accurately. This process can be continued until self-consistency is achieved, i.e. until the difference between two successive values of ξ is negligible. Practically this is a complicated method and is difficult to isolate and determine the roles of the physical aspects of the problem, such as the effects of the exclusion principle and the motion of the centre of mass.

To simplify the mathematics we introduce the method of reference spectrum of Bethe et al⁽⁴⁾. Eq. (3.2) and (3.3) are solved under the following conditions: (a) The region of integration over K' is extended to include the Fermi surface; (b) we assume that the centre of mass of the two

interacting particles is at rest, i.e. $p = 0$; (c) we fit $e(K, K')$ with a quadratic form as a function of K' , $e(K, K') = [K'^2 + \gamma^2(K)]/m^*$ where m^* is the effective mass. In (c) we have assumed that e remains a negative quantity for all values of K and K' and this implies that no real scattering can take place in Fermi sea. The resulting partial differential equation is

$$(\nabla^2 - \gamma^2)[\psi(\vec{r}) - \exp(i\vec{K} \cdot \vec{r})] = m^*V(r) \theta(\xi - r)\psi(r) \quad (3.4)$$

In the case of a central potential we can use the partial wave decomposition of $\psi(\vec{r})$

$$\psi(\vec{r}) = \frac{1}{r} \sum_{\ell=0}^{\infty} (2\ell+1)i^\ell P_\ell(\cos \theta) u_\ell(r) \quad (3.5)$$

to reduce equation (3.4) to a set of total differential equations:

$$\left[\frac{d}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \gamma^2 \right] [u_\ell(r) - r j_\ell(Kr)] = m^* V^S(r) u_\ell(r) \quad (3.6)$$

with the boundary conditions $u_\ell(c) = 0$ and $\quad (3.7)$

$$\left[\frac{d(u_\ell(r) - r j_\ell(Kr))/dr}{u_\ell(r) - r j_\ell(Kr)} \right]_{r=\xi} = \left[\frac{d(r h_\ell^{(1)}(i\gamma r))/dr}{r h_\ell^{(1)}(i\gamma r)} \right]_{r=\xi} \quad (3.8)$$

where $h_\ell^{(1)}$ is the spherical Hankel function of the first kind.

The separation distance is defined by equation (3.3) which can be expressed in partial waves as follows:

$$\frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \int_0^{\xi(K)} r j_\ell(Kr) V(r) u_\ell(r) dr = 0 \quad (3.9)$$

Substituting the integrand for $V(r)u_\ell(r)$ from (3.6), we find

$$\begin{aligned} & \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \int_0^{\xi(K)} r j_\ell(Kr) V(r) u_\ell(r) dr \\ &= \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \frac{1}{m^*} \int_0^{\xi(K)} r j_\ell(Kr) \left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \gamma^2 \right) \\ & \quad (u_\ell - r j_\ell(Kr)) dr \\ &= -\frac{\pi}{2m^*} \sum_{J,T} (2J+1)(2T+1) \left\{ (\gamma^2 + K^2) \int_0^\xi r j_\ell(Kr) (r j_\ell(Kr) - u_\ell(r)) dr \right. \\ & \quad + \left[r j_\ell(Kr) \frac{d}{dr} \{ u_\ell(r) - r j_\ell(Kr) \} - \{ u_\ell(r) - r j_\ell(Kr) \} \right. \\ & \quad \left. \left. \frac{d}{dr} r j_\ell(Kr) \right] \right\}_{r=\xi} = 0. \quad (3.10) \end{aligned}$$

The last step involves integration by parts and substitution of the boundary condition (3.8). As mentioned before, at the region $r > \xi_0$ it is good approximation to replace the exact wave function by the plane wave. In doing so the last term of eq. (3.10) can be neglected.

Then (3.10) becomes

$$\begin{aligned} & \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \int_0^{\xi(K)} r j_\ell(Kr) V(r) u_\ell(r) dr \\ &= \frac{\pi}{2m^*} \sum_{J,T} (2J+1)(2T+1)(\gamma^2 + K^2) \int_0^{\xi(K)} r j_\ell(Kr) \chi_\ell(Kr) dr \end{aligned} \quad (3.11)$$

where $\chi_\ell(Kr) = r j_\ell(Kr) - u_\ell(r)$.

Using the approximation $u_\ell(r) \cong r j_\ell(Kr)$ for $r > \xi_0$ in right side of (3.11) again we get

$$\begin{aligned} & \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \left[\frac{\gamma^2 + K^2}{m^*} \int_0^{\xi_0(K)} r j_\ell(Kr) \chi_\ell(Kr) dr + \right. \\ & \quad \left. \int_{\xi_0}^{\xi} r^2 j_\ell^2(Kr) V(r) dr \right] = 0. \end{aligned} \quad (3.12)$$

For hard core potential the exact wave function inside the core is zero, i.e. $u_\ell(r) = 0$ for $r < c$, so that (3.12) can be written as

$$\begin{aligned} & \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \frac{\gamma^2 + K^2}{m^*} \left[\int_0^c r^2 j_\ell^2(Kr) dr + \int_c^{\xi} r^2 j_\ell(Kr) \chi_\ell(r) dr \right] \\ &= \frac{\pi}{2} \sum_{J,T} (2J+1)(2T+1) \int_{\xi_0}^{\xi} r^2 j_\ell^2(Kr) V(r) dr. \end{aligned} \quad (3.14)$$

The quantity $\gamma^2(K)$ is related to the single-particle potential energy $u(K_i)$, which can be written as quadratic function of k_i .

$$u(k_i) = \left(\frac{1}{m^*} - 1 \right) \frac{K_i^2}{2} + \Delta - \frac{a+bK_i^2}{1+cK_i^2} \frac{\theta(K_f - K_i)}{2} \quad (3.15)$$

where Δ , m^* , a , b and c are constants, and θ is the step function, then the energy denominator can be written as:

$$e(K, K') = \frac{K^2 - K'^2}{m^*} - \frac{a + b K^2}{1 + c K^2} \quad (3.16)$$

and thus
$$\gamma^2(K) = \frac{m^*(a + b K^2)}{1 + c K^2} - K^2. \quad (3.17)$$

Using the first order approximation ξ_0 to calculate the single-particle potential energy $u(K_i)$ and fitting in (3.15), (3.16) and (3.17) we can have the first approximation to the energy denominator $e(K, K')$ and the effective mass m^* . Then we calculate x_ℓ by solving eq. (3.6) and integrate (3.14) to find a more accurate separation distance ξ . This process can be continued until self-consistency is achieved.

For the particle above the Fermi Sea the contribution of odd states is very small and can be neglected⁽²⁹⁾. Thus we can use the same process used for particles inside the Fermi sea to calculate single-particle energy above the Fermi sea but only take the even states with statistical weight 1 instead of 3/4 into account. After calculating the first order separation distance and hence the first order potential energy, the second order approximation can be calculated easily.

Once the exact separation distance is found, the

contribution of the long range potential may be obtained from the Born series. Thus

$$V_{\text{exact}}^{\ell} = V^{\ell} + V^{\ell} \frac{Q}{e} V^{\ell} + \dots \quad (3.18)$$

If instead of the plane wave approximation we were using the unperturbed wave function $|\phi\rangle$, we could write

$$V_{\text{exact}}^{\ell} = \langle e^{i\vec{K}\cdot\vec{r}} | V^{\ell} | \phi_0 \rangle \quad (3.19)$$

where the integral equation for $|\phi_0\rangle$ is given by

$$\phi_0(Kr) = e^{i\vec{K}\cdot\vec{r}} + \int dr' e^{i\vec{K}\cdot(\vec{r}-\vec{r}')} \frac{Q}{e} V^{\ell}(r') \phi_0(Kr') \quad (3.20)$$

By successive iteration of (3.20) and substitution in (3.19) we find (3.18). There will be some scattering to states outside the Fermi sea. The contribution from this scattering is expected to be quite small for the following reasons: (1) V^{ℓ} is already small, and (2) the Pauli principle strongly inhibits any scattering caused by V^{ℓ} . The diagonal elements of $V^{\ell} \frac{Q}{e} V^{\ell}$ for each state can be written explicitly as

$$\langle K | V^{\ell} \frac{Q}{e} V^{\ell} | K \rangle_{\ell, J}^{S, T} = \int_0^{\infty} \frac{\langle K | V^{\ell} | K' \rangle_{\ell, J}^{S, T} \langle K' | V^{\ell} | K \rangle_{\ell, J}^{S, T} Q(p, K')}{e(K, K', p)} \frac{4\pi K'^2}{(2\pi)^3} dK'$$

$$\begin{aligned}
&= \int_0^\infty \frac{Q(p, K')}{e(K, K', p)} \frac{4\pi K'^2}{(2\pi)^3} dK' \left[\int_\xi^\infty j_\ell(K'r) V(r)_{\ell, J}^{S, T} j_\ell(Kr) r^2 dr \right]^2 \\
&= 8 \int_0^\infty \frac{Q(p, K')}{e(K, K', p)} K'^2 dK' \left[\int_\xi^\infty j_\ell(K'r) V(r)_{\ell, J}^{S, T} j_\ell(Kr) r^2 dr \right]^2
\end{aligned} \tag{3.21}$$

then for all partial waves this contribution is

$$\begin{aligned}
\langle K | V^\ell \frac{Q}{e} V^\ell | K \rangle &= \frac{1}{8} \sum_{J, T} (2J+1)(2T+1) \langle K | V^\ell \frac{Q(p, K')}{e(K, K', p)} V^\ell | K \rangle_{\ell, J}^{S, T} \\
&= \sum_{J, T} (2J+1)(2T+1) \int_0^\infty \frac{Q(p, K')}{e(K, K', p)} K'^2 dK' \\
&\quad \left[\int_\xi^\infty j_\ell(K'r) V_{\ell, J}^{S, T} j_\ell(Kr) r^2 dr \right]^2
\end{aligned} \tag{3.23}$$

where p is the centre-of-mass momentum of the interacting pair.

The effect of p is greatest in the exclusion operator Q and is not so important in the energy denominator. For this reason we replace p in Q by the average (over the Fermi sea) centre - of - mass momentum of two particles having a relative momentum K . Thus

$$\begin{aligned}
\frac{p^2}{4} &= \frac{3}{5} K_f (K_f - K) \left[1 + \frac{K^2}{3K_f(2K_f + K)} \right] \text{ for } K < K_f \\
&= 0. \qquad \qquad \qquad K > K_f
\end{aligned} \tag{3.23}$$

A cruder approximation was used in the energy denominator where the value $p = K_f$ was used. This value corresponds to the peak in the distribution of the number of pairs of particles having centre - of - mass momentum p ⁽⁵⁾.

Chapter 4.

Numerical Calculation with BGT Potential

A. Details of Calculation and Results

The Method described in the previous chapters has been used to calculate the effective interaction for the Brueckner - Gammel - Thaler (BGT) potential. The BGT potential contains a central, a spin - orbit and a tensor term and has Yukawa well outside a hard core .

$$\begin{aligned}
 V(r) &= +\infty & r < c \\
 V(r) &= V_c \frac{e^{-\mu_c r}}{\mu_c r} + \vec{L} \cdot \vec{S} V_{LS} \frac{e^{-\mu_{LS} r}}{\mu_{LS} r} + \vec{S}_{12} V_T \frac{e^{-\mu_T r}}{\mu_T r} & r > c
 \end{aligned}
 \tag{4.1}$$

The parameters of this potential are shown in Table I.

TABLE I.

Parameters of the BGT Potential

The Core Radius is 0.4 Fermi

| | SPIN | PARITY | V(MeV) | $\mu (F^{-1})$ |
|------------|---------|--------|--------|----------------|
| Central | Singlet | even | -434 | 1.45 |
| | singlet | odd | +130 | 1.00 |
| | triplet | even | -877.4 | 2.091 |
| | triplet | odd | -14.0 | 1.00 |
| Spin-Orbit | Triplet | even | -5000 | 3.70 |
| | triplet | odd | -7315 | 3.70 |
| Tensor | triplet | even | -159.4 | 1.045 |
| | triplet | odd | +22.0 | 0.08 |

TABLE II.

Phase Shifts Calculated from the Phase FunctionsThe Phase Shifts are in radians

| $K_f(F^{-1})$ | $E(\text{MeV})$ | 1S_0 | 1P_1 | 1D_2 | 3P_0 | 3P_1 | 3P_2 | 3D_2 |
|---------------|-----------------|---------|---------|---------|---------|---------|---------|---------|
| 1.1 | 30 | .789 | -.14 | .0154 | .248 | -.112 | .08 | .0764 |
| 1.2 | 35.8 | .741 | -.16 | .021 | .26 | -.12 | .09 | .10 |
| 1.3 | 42 | .693 | -.18 | .027 | .269 | -.14 | .11 | .128 |
| 1.4 | 48.7 | .646 | -.20 | .0358 | .271 | -.15 | .13 | .158 |
| 1.5 | 55.9 | .6 | -.218 | .0446 | .27 | -.17 | .148 | .19 |
| 1.6 | 63.6 | .55 | -.236 | .054 | .26 | -.18 | .167 | .224 |
| 1.7 | 71.9 | .511 | -.25 | .0649 | .255 | -.198 | .185 | .259 |

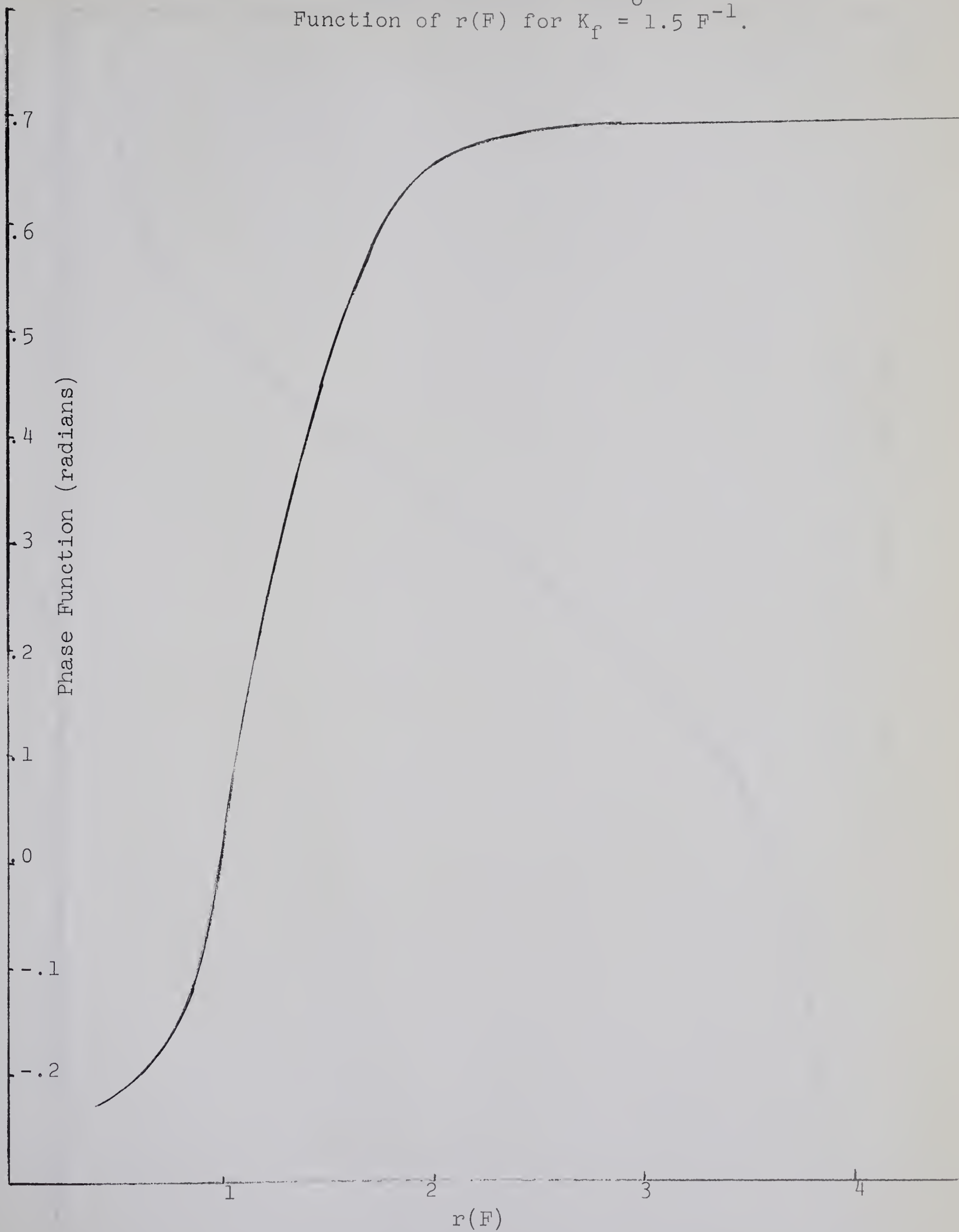
TABLE III.

Phase Shifts Calculated by Gammel & Thaler from 4205 Potential

The Phase shifts are in radians

| E(MeV) | 1S_0 | 1D_2 | 3P_0 | 3P_1 | 3P_2 |
|--------|---------|---------|---------|---------|---------|
| 30 | .771 | .014 | .219 | -.121 | .024 |
| 40 | .695 | .024 | .2365 | -.149 | .0995 |
| 60 | .5068 | .047 | .2365 | -.194 | .14555 |
| 80 | .466 | .0725 | .213 | -.231 | .1835 |

Fig.1a - The Phase Function of 1S_0 State as a Function of $r(F)$ for $K_F = 1.5 F^{-1}$.



Phase Function (radians)

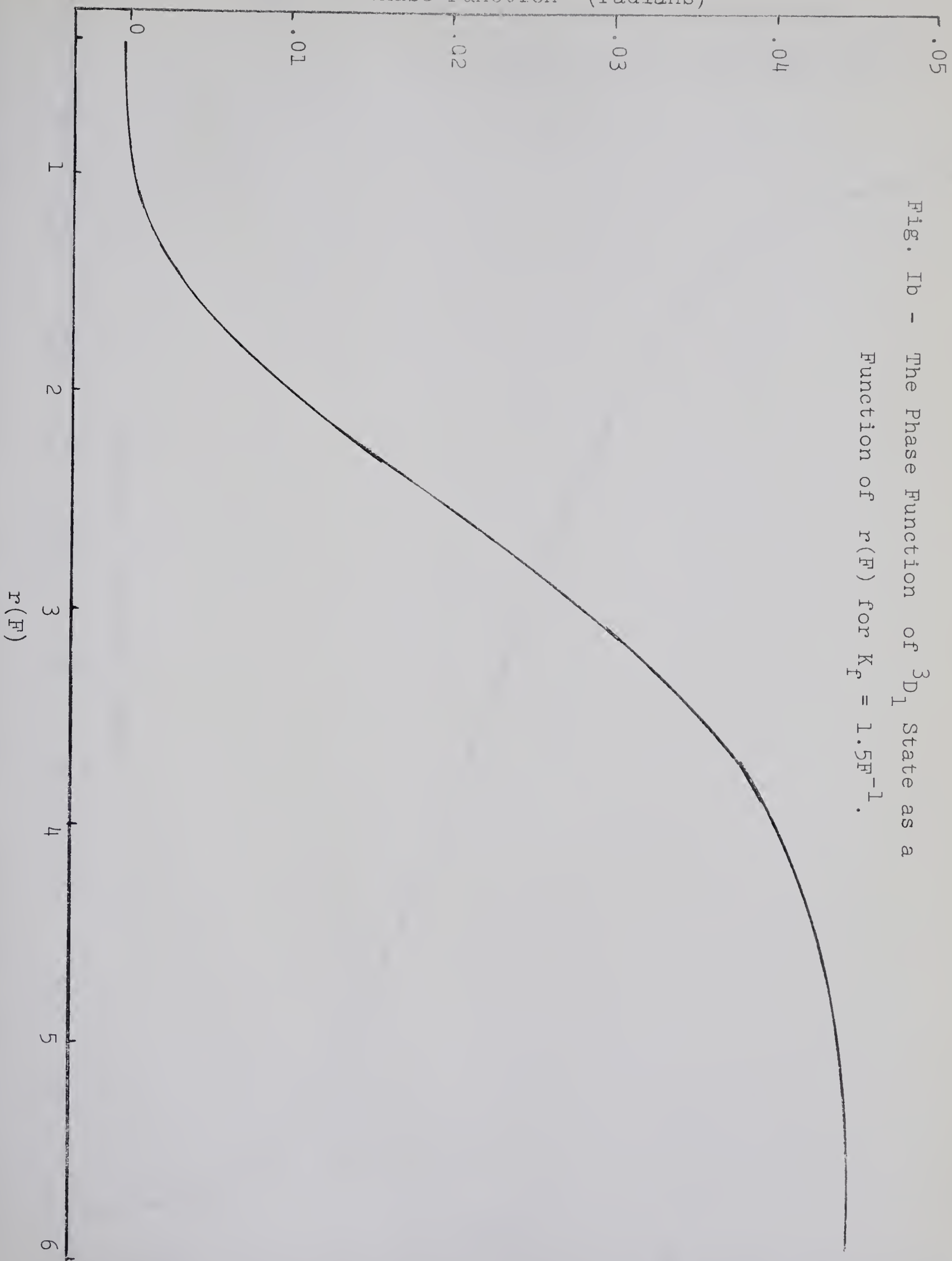


Fig. Ib - The Phase Function of 3D_1 State as a Function of $r(F)$ for $K_F = 1.5F^{-1}$.

Fig. 1c - The Phase Function of 3D_2 State
as a Function of $r(F)$ for $K_f = 1.5F^{-1}$.

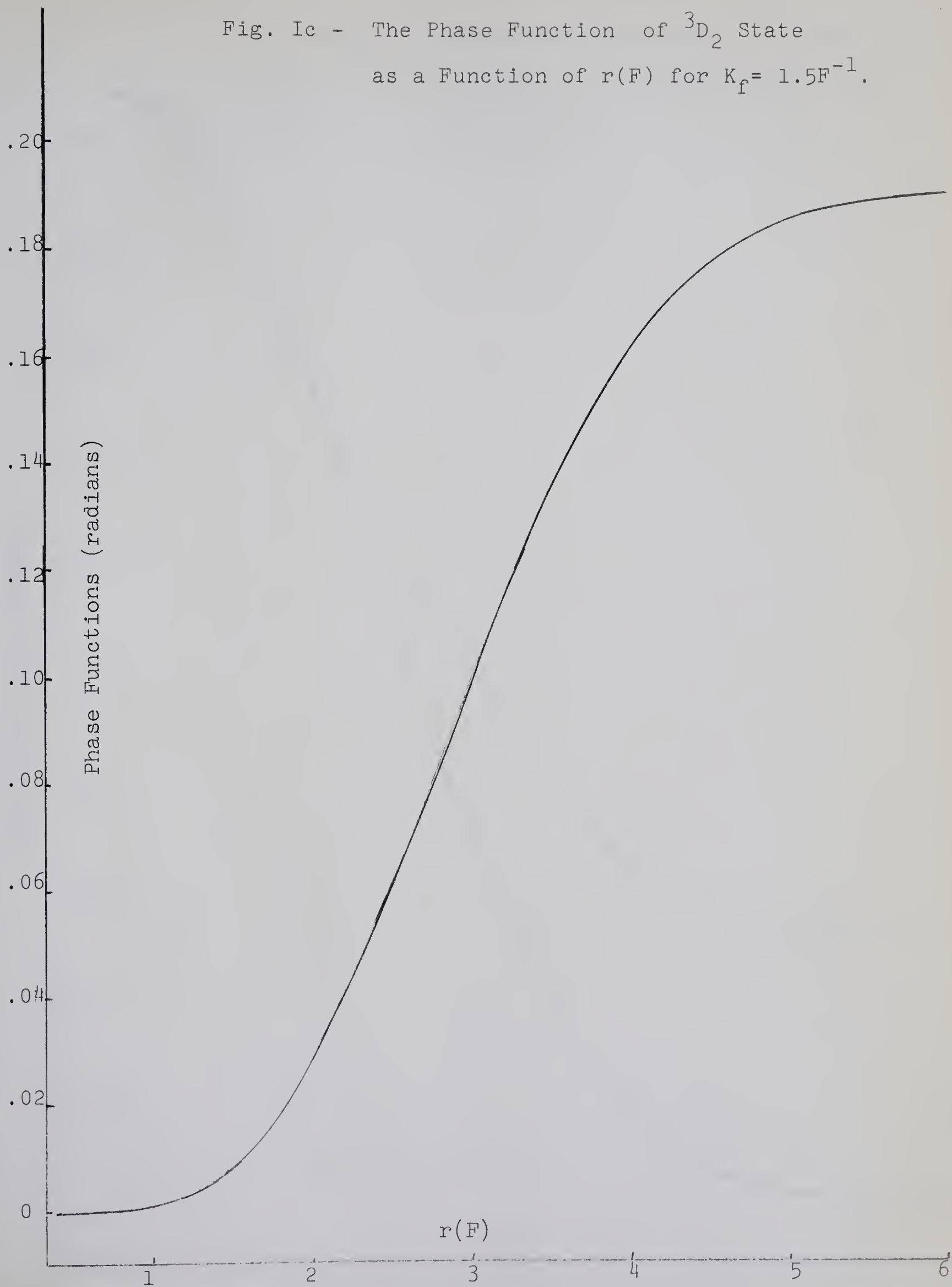
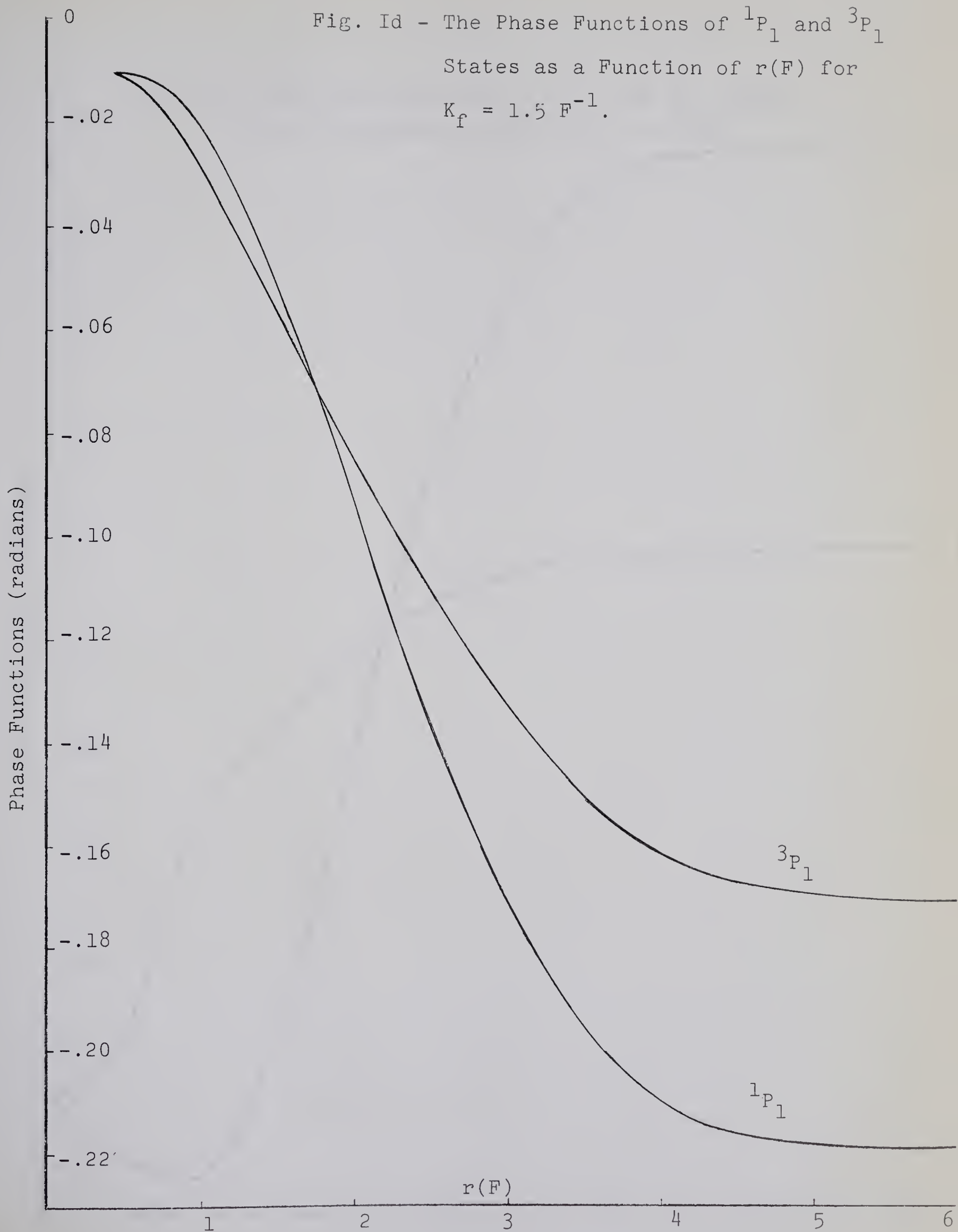
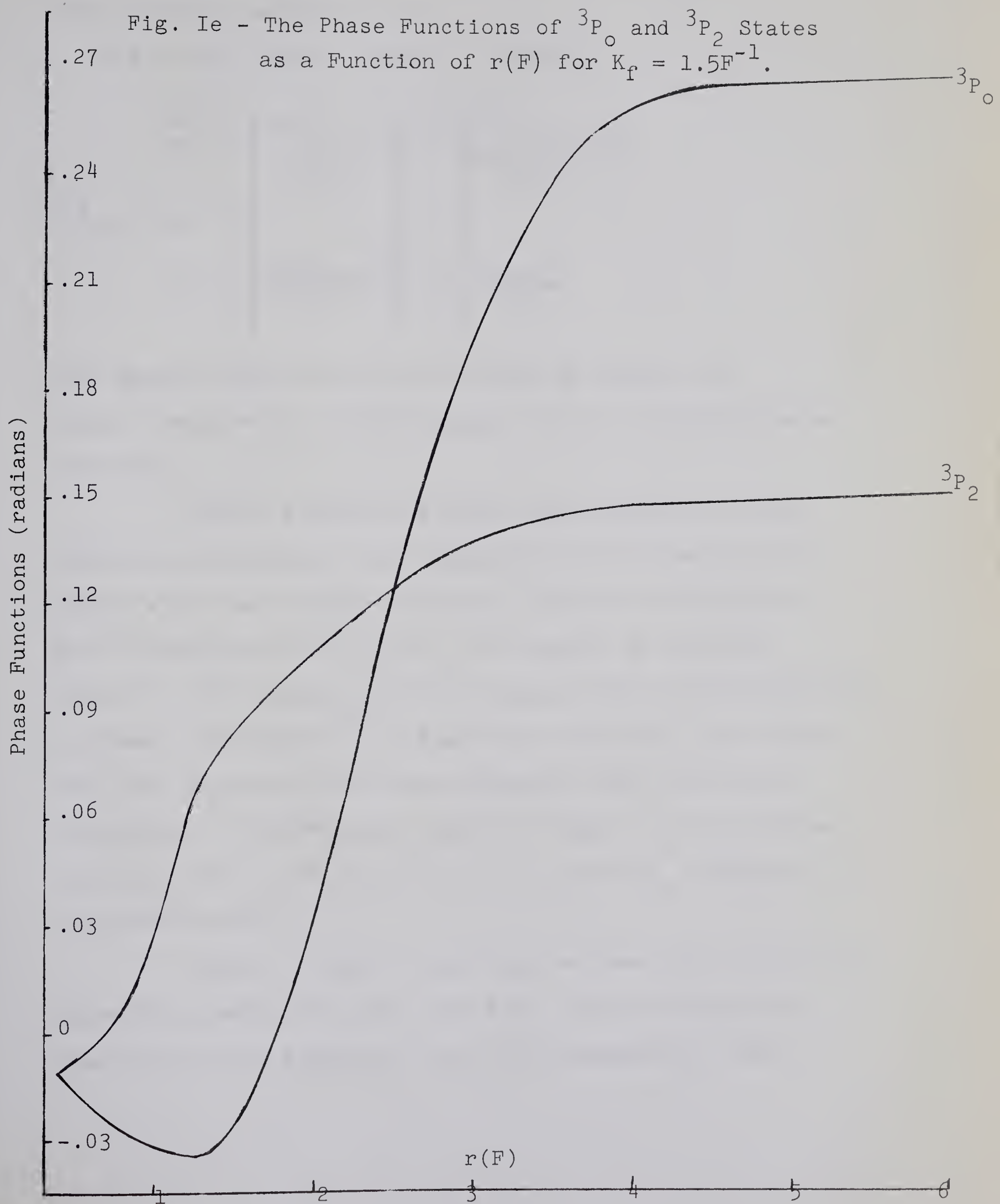


Fig. 1d - The Phase Functions of 1P_1 and 3P_1
States as a Function of $r(F)$ for
 $K_f = 1.5 F^{-1}$.





The spin-orbit and tensor operators $\vec{L} \cdot \vec{S}$ and \vec{S}_{12} have their standard meaning, i.e.

$$\vec{L} \cdot \vec{S} = \frac{1}{2} [J(J+1) - L(L+1) - S(S+1)] \quad (4.2)$$

$$V_T \vec{S}_{12} = V_T(r) \cdot \begin{array}{c|ccc} L'L'' & J-1 & J & J+1 \\ \hline J-1 & \frac{-2(J-1)}{2J+1} & 0 & \frac{6[J(J+1)]^{1/2}}{2J+1} \\ J & 0 & 2 & 0 \\ J+1 & \frac{6[J(J+1)]^{1/2}}{2J+1} & 0 & \frac{-2(J+2)}{2J+1} \end{array}$$

This matrix relation can be obtained by acting the tensor operator \vec{S}_{12} on the angular and spin dependent wave functions.

Before finding the first approximation to the separation distance, ξ_0 , we integrate (2.30) from the core radius with the boundary condition (2.32) to infinity to get the phase shift, $\delta_{\ell, J}^{S, T}$ (K), the result is shown in Table II. The result has been compared with those calculated by Gammel and Thaler⁽²⁵⁾ using GT4205 potential (See Table III), and shows the excellent agreement with the direct integration of Schrödinger equation. Figs. 1 are the phase functions for $^1S_0, ^1P_1, ^3P_0, ^3P_1, ^3P_2, ^1D_2$, and 3D_2 states as functions of $r(F)$.

In all of our calculations we used the fourth-order Runge-Kutta method to solve the first order differential equation and the Simpson's rule for integration. Due to

the strong coupling by the tensor force we calculate the separation distance for 3S_1 state by solving the coupled differential equation (2.35) and (2.36) as stated in Chapter 3. The coupling between 3P_1 and 3F_1 is small, so that we neglect the coupling in 3P_1 state. We also calculate the 3D_3 state separately because its separation distance is close to the core radius and the contribution of 3D_3 to the potential energy is small. Then it is easy to integrate the phase function (2.30) for all the states $l \leq 2$ except 3S_1 , 3D_1 and 3D_3 starting from the core radius to the point where equation (2.32) is satisfied. Moszkowski et al⁽⁵⁾ have pointed out that the separation distance depends upon the momentum and also on the angular momentum. However, in our calculation, we introduce only one separation distance for all partial waves (except those mentioned above). The result of our calculation shows that the approximate separation distance is nearly independent of K_f and K , and it has a value of $1.07F$ ($1F=10^{-13}\text{cm}$). This result (i.e. ξ_0 independent of K) may be related to the fact that the cross-section for nucleon-nucleon scattering remains constant over a wide range of energies. After finding the separation distance, the potential energy can be easily calculated from (2.16) and (2.21) by using numerical integration.

To calculate the separation distance for 3S_1 state, we factorize the second order differential (2.41) and (2.42) into the following six first order differential equation⁽³⁾.

$$\frac{dY_1}{dr} = - [1 + f(r) Y_1^2] \quad (4.4)$$

$$\frac{dY_2}{dr} = - Y_1 [f(r) Y_2 + V_{11}\phi - V_{12} \zeta] \quad (4.5)$$

$$\frac{dY_3}{dr} = f(r) [Y_3 Y_1 + Y_2] + [V_{11}\phi - V_{12} \zeta] \quad (4.6)$$

$$\frac{dZ_1}{dr} = -[1 + g(r) Z_1^2] \quad (4.7)$$

$$\frac{dZ_2}{dr} = -Z_1 [V_{12}(\phi - x) + g(r) Z_2] \quad (4.8)$$

$$\frac{dZ_3}{dr} = g(r) [Z_3 Z_1 + Z_2] + V_{12} (\phi - x) \quad (4.9)$$

$$\text{Here } f(r) = K^2 - V_{11} \quad (4.10)$$

$$g(r) = K^2 - \frac{6}{r^2} - V_{22} \quad (4.11)$$

$$\phi(r) = r j_0(Kr) \quad (4.12)$$

$$x = Y_1 Y_3 + Y_2 \quad (4.13)$$

$$\text{And } \zeta = Z_1 Z_3 + Z_2 \quad (4.14)$$

Equation (4.4) and (4.5) are subject to the following boundary conditions:

$$Y_1(c) = 0 \quad (4.15)$$

$$Y_2(c) = \phi(c) \quad (4.16)$$

The point where $Y_2(r)$ is zero is the desired point ξ_0 , i.e.

$$Y_2(\xi_0) = 0 \quad (4.17)$$

At this point we put

$$Y_3(\xi_0) = 0 \quad (4.18)$$

Since χ and χ' are continuous functions of r , it follows that

$$\chi(\xi_0) = \chi'(\xi_0) = 0 \quad (4.19)$$

Assuming the continuity of ζ at $r = \xi_0$, we have

$$\frac{\zeta'(\xi_0)}{\zeta(\xi_0)} = \left[\frac{d[rj_2(Kr)]}{dr} \bigg/ rj_2(Kr) \right]_{r=\xi_0} \quad (4.20)$$

To start integration we use the trial function $\zeta = -rj_2(Kr)$ and integrate (4.4) and (4.5) from the core radius with the boundary conditions (4.15) and (4.16) outward to the point ξ_0 satisfying (4.17), then with the boundary condition (4.18) we integrate (4.6) inward to c and calculate χ from (4.13). Knowing χ we are able to integrate the equation for Z_1 and Z_2 from c to ξ_0 with the boundary conditions

$$Z_1(c) = 0 \quad (4.21)$$

$$Z_2(c) = 0 \quad (4.22)$$

From equation (4.7), (4.8), (4.9), (4.14) and (4.19) it follows that

$$Z_3(\xi_0) = - \frac{\left[Z_2(\xi_0) \left(\frac{d[rj_2(Kr)]}{dr} \right)_{r=\xi_0} \right]}{\left[rj_2(Kr) + Z_1(\xi_0) \frac{d[rj_2(Kr)]}{dr} \right]_{r=\xi_0}} \quad (4.23)$$

With this value of Z_3 we can integrate (4.8) from ξ_0 to c and find ζ from (4.14). Usually two or three iterations are enough to give an accurate value of ξ_0 , x and ζ .

The separation distances for 3S_1 state are given in Table IV. It is interesting to note that the separation distance for 3S_1 does not depend strongly on K at low energies.

The potential energy for 3S_1 state can be calculated as for $^1S_0 \sim ^3D_2$ states. The separation distance for 3D_1 and 3D_3 are very close to core radius, thus we use the following approximation to calculate the potential energies:

$$\bar{U}_{^3D_J} = \frac{2K_f^3}{3\pi^2} \cdot \frac{\pi}{2} (2J+1) \delta_l^J(K, \infty) \quad (4.24)$$

The potential energies for all of the states are given in Table V.

TABLE IV.

The Separation Distances for 3S_1 State

| $K_f(F^{-1})$ | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 |
|---------------|-----|-----|-----|------|------|------|------|
| $\xi_0(F)$ | .97 | .98 | .99 | 1.01 | 1.02 | 1.05 | 1.09 |

TABLE V.

The Contribution of Different Waves to the Average

Energy per Particle. All Unmarked Units are in MeV.

| $K_F(F^{-1})$ | $1S_0$ | $3S_1$ | $1P_1$ | $3P_0$ | $3P_1$ | $3P_2$ | $1D_2$ | $3D_1$ | $3D_2$ | $3D_3$ | TOTAL |
|---------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|---------|
| 1.1 | -9.68 | -5.81 | 2.35 | -3.37 | 5.08 | -3.99 | -1.028 | .2772 | -1.5 | -.433 | -17.876 |
| 1.2 | -11.5 | -6.984 | 3.17 | -4.36 | 6.75 | -5.46 | -1.667 | .3652 | -2.32 | -.641 | -22.354 |
| 1.3 | -13.8 | -8.177 | 4.14 | -5.47 | 8.7 | -7.24 | -2.565 | .6048 | -3.41 | -.895 | -26.84 |
| 1.4 | -15.3 | -9.56 | 5.28 | -6.66 | 10.9 | -9.35 | -3.775 | .840 | -4.8 | -1.187 | -32.619 |
| 1.5 | -17.1 | -10.26 | 6.47 | -7.93 | 13.46 | -11.8 | -5.35 | 1.12 | -6.5 | -1.5 | -38.338 |
| 1.6 | -18.8 | -11.94 | 7.8 | -9.2 | 16.2 | -14.6 | -7.34 | 1.4685 | -8.6 | -1.84 | -45.387 |
| 1.7 | -20.3 | -10.81 | 9.29 | -10.5 | 19.36 | -17.7 | -9.8 | 1.865 | -11.08 | -2.17 | -53.09 |

To calculate first and second order corrections we have to calculate also the potential energy above the Fermi sea and fit the potential energy in and above Fermi sea into equation (3.15). Then from (3.15), (3.16) and (3.17) we can get the values of the effective mass, energy denominator and $\gamma^2(K)$. The single particle potential energy as function of relative momentum for $K_f = 1.5F^{-1}$ is shown in Fig 2. The values of the constant for equation (3.15) are shown in Table VI.

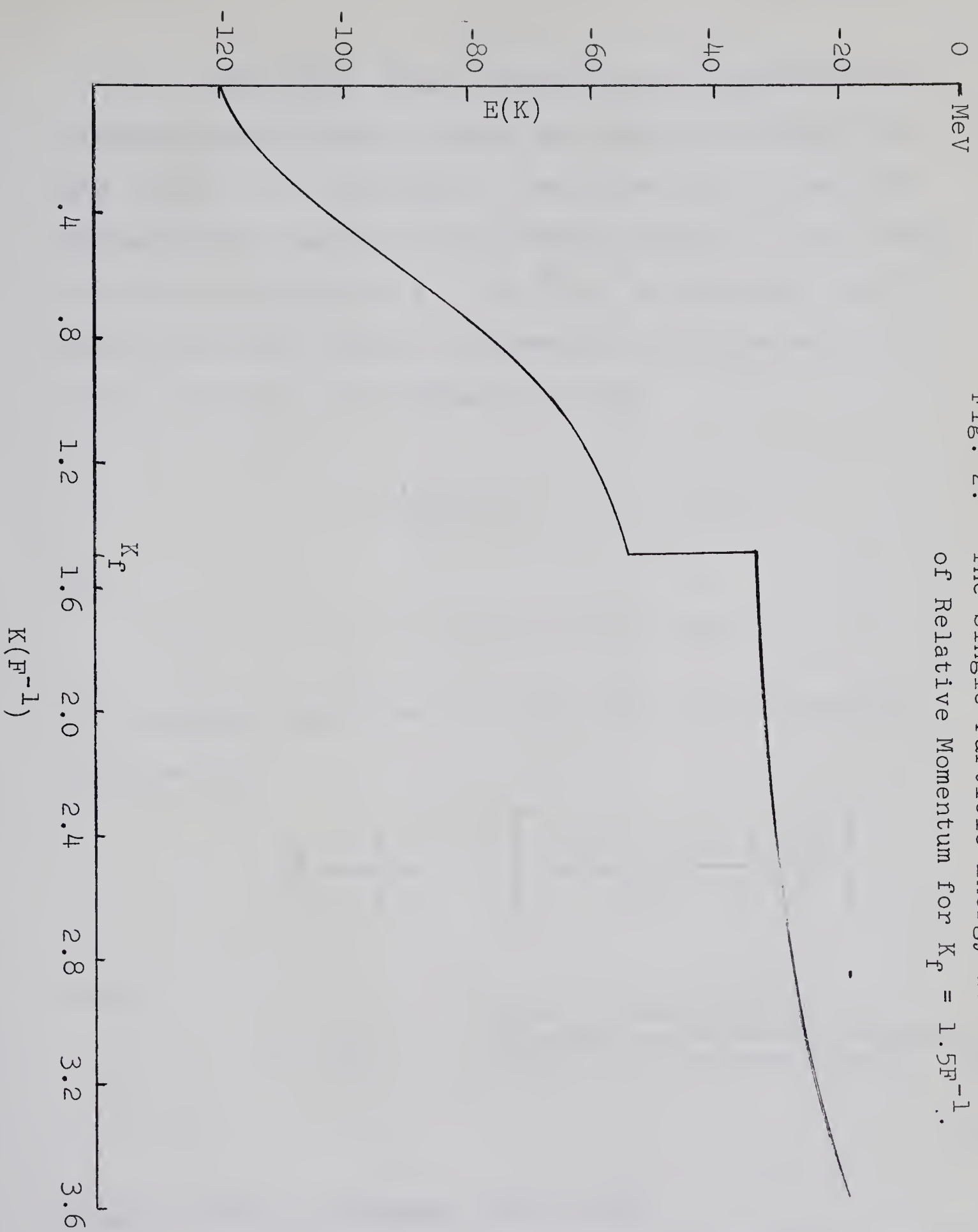
TABLE VI.

The Values of Effective Mass, $\gamma^2(K)$ and the Constants
for Energy Denominator

| $K_f(F^{-1})$ | $\gamma^2(F^{-2})$ | m^* | $a(F^{-2})$ | b | $c(F^2)$ |
|---------------|--------------------|--------|-------------|---------|----------|
| 1.4 | 1.074 | .95621 | 3.25447 | -.01132 | 1.5476 |
| 1.5 | 1.1739 | .94666 | 4.00285 | -.01393 | 1.5476 |
| 1.6 | 1.4206 | .93603 | 4.85798 | -.0169 | 1.5476 |

m^* and b are dimensionless numbers.

Fig. 2. - The Single Particle Energy as a Function of Relative Momentum for $K_F = 1.5F^{-1}$.



The first order corrections to the separation distance and potential energy are calculated from (3.6) and (3.14). To calculate x_ℓ we solve the second order differential equation by the method similar to that used in the calculation of ξ_0 for 3S_1 . We factorize (3.6) into three first order differential equations as (4.4), (4.5) and (4.6), but replace $f(r)$ by

$$- \left(\frac{\ell(\ell + 1)}{r^2} + \gamma^2 + m^* V \right)$$

and

$$V_{11} \phi - V_{12} \zeta \text{ by } -m^* V r j_\ell(Kr).$$

The boundary conditions of (2.45) and (2.46) are also replaced by

$$\frac{x_\ell(\xi_0)}{x_\ell(\xi_0)} = \left[\frac{d[rh_\ell^{(1)}(i\gamma r)]/dr}{rh_\ell^{(1)}(i\gamma r)} \right]_{r=\xi_0} \quad (4.25)$$

and

$$Y_3(\xi_0) = - \left[\frac{\{d[rh_\ell^{(1)}(i\gamma r)]/dr\} Y_2}{\{d[rh_\ell^{(1)}(i\gamma r)]/dr\} Y_1 + rh_\ell^{(1)}(i\gamma r)} \right]_{r=\xi_0} \quad (4.26)$$

where (4.26) is obtained from (4.25).

The first order correction of separation distances and contribution potential energies are given in Table VII.

TABLE VII

The First Order Correction of Separation Distances,
 $\xi(F^{-1})$, and Contribution Potential Energies(MeV)

| $K_F(F^{-1})$ | $\xi^1 S_0 \sim^3 D_2$ | $\xi^3 S_1$ | $^1 S_0$ | $^3 S_1$ | $^1 P_1$ | $^3 P_0$ | $^3 P_1$ | $^3 P_2$ | $^1 D_2$ | $^3 D_2$ | TOTAL |
|---------------|------------------------|-------------|----------|----------|----------|----------|----------|----------|----------|----------|-------|
| 1.4 | 1.102 | 1.096 | 2.195 | 1.4335 | .36 | 1.084 | 3.77 | 1.96 | -7.354 | -1.67 | 1.779 |
| 1.5 | 1.166 | 1.144 | 3.2 | 1.89 | .49 | 1.875 | 6.37 | 2.812 | -9.59 | -2.202 | 4.86 |
| 1.6 | 1.284 | 1.232 | 4.688 | 3.47 | .77 | 4.615 | 20.95 | 4.26 | -11.8 | -2.83 | 24.1 |

The second order potential terms can be obtained by integrating (3.22) with corrected separation distance ξ . The result is given in Table VIII.

TABLE VIII.

The Contribution of the Second Order Terms to the Potential Energy. All unmarked Units are in MeV.

| $K_f(F^{-1})$ | 1S_0 | 3S_1 | 1P_1 | 3P_0 | 3P_1 | 3P_2 | 1D_2 | 3D_2 | TOTAL |
|---------------|---------|---------|---------|---------|---------|---------|---------|---------|-------|
| 1.4 | -.143 | -.05 | -.267 | -.024 | -.656 | -.689 | -.295 | -.484 | -2.61 |
| 1.5 | -.2 | -.07 | -.209 | -.033 | -.61 | -.684 | -.34 | -.525 | -2.67 |
| 1.6 | -.454 | -.14 | -.151 | -.06 | -.527 | -.635 | -.375 | -.544 | -2.88 |

The binding energies per particle in nuclear matter is given in Table IX.

TABLE IX.

The Total Energy per Particle (MeV) Versus Fermi Momentum

53.

| $k_F(F^{-1})$ | $1S_0$ | $3S_1$ | $1P_1$ | $3P_0$ | $3P_1$ | $3P_2$ | $1D_2$ | $3D_1$ | $3D_2$ | $3D_3$ | $\frac{P.E.}{A}$ | $\frac{K.E.}{A}$ | $\frac{B.E.}{A}$ |
|---------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|------------------|------------------|------------------|
| 1.4 | -13.2 | -8.1 | 5.3 | -5.6 | 14.0 | -8.0 | -11.4 | .84 | -8.1 | -1.1 | -33.9 | 24.3 | -9.1 |
| 1.5 | -13.9 | -8.44 | 6.7 | -6.0 | 19.5 | -9.6 | -15.2 | 1.12 | -9.2 | -1.5 | -35.1 | 27.9 | -7.2 |
| 1.6 | -14.56 | -8.6 | 8.4 | -4.6 | 36.5 | -10.9 | -18.7 | 1.4 | -11.9 | -1.8 | -24.1 | 31.8 | 6.6 |

B. Discussion.

The calculations of Brueckner and Masterson⁽²⁾(BM) and of Razavy both give about 8 MeV per particle binding energy for equilibrium density with $k_f = 1.28$ and 1.12f^{-1} respectively. G.E. Brown et al ⁽²⁸⁾ have investigated some possible causes of this discrepancy very thoroughly. They found that in both calculations the binding energy was overestimated, making the discrepancy worse. In BM calculation, an angle average was used and the average total momentum was inserted in the exclusion principle. However, this is not a good approximation and causes an error of about 1.7 MeV. In addition, BM calculation is approximate because of incorrect treatment of the off-energy effects in intermediate states. They neglected the effective mass introduced by off-energy shell propagation which causes an error of 2-4 MeV/particle. Razavy, on the other hand, overestimated the binding energy by omitting the Pauli and spectral corrections. Correcting for this would lower the binding energy by about the same amount as that of Brueckner and Masterson.

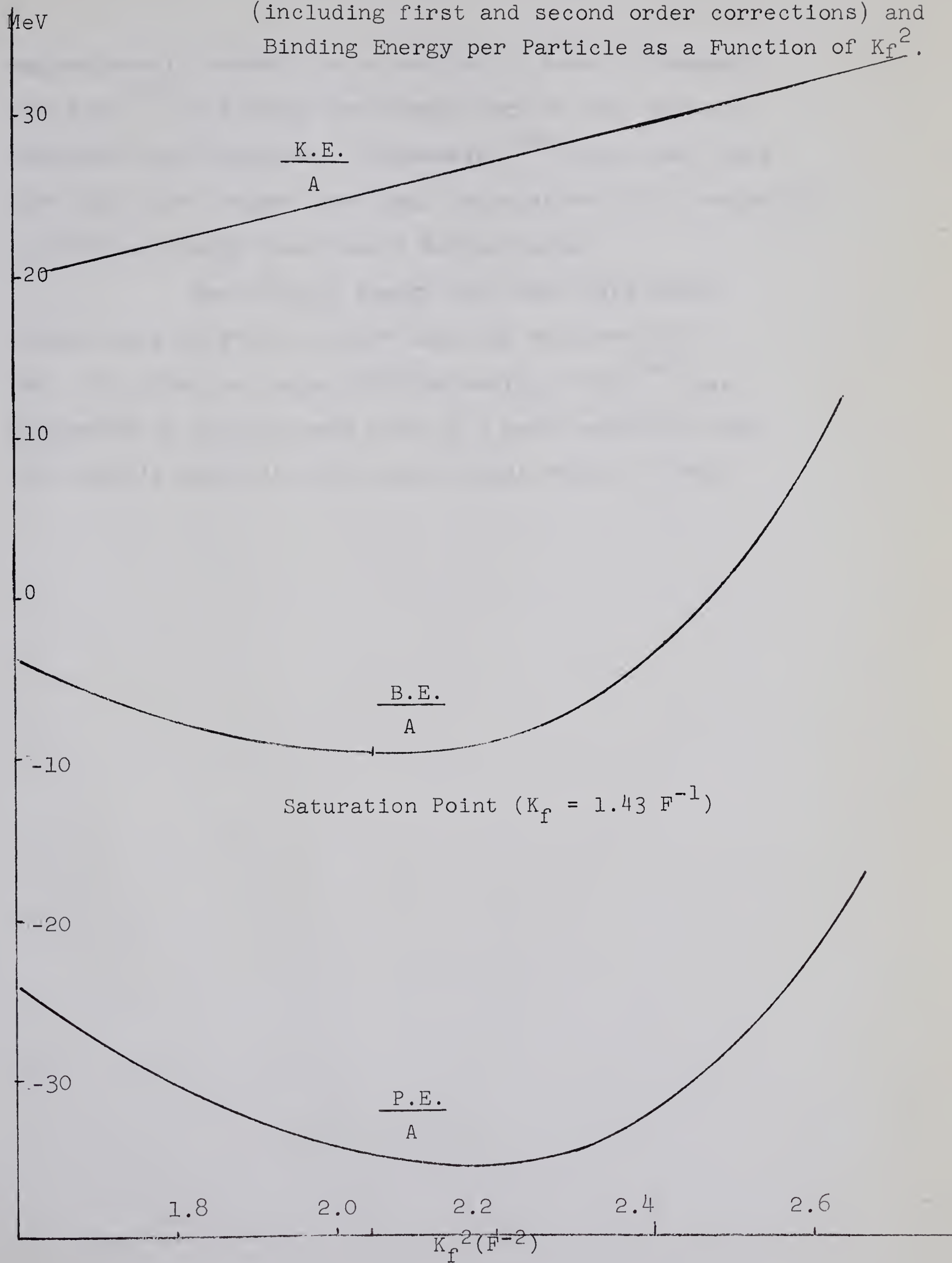
In our method, by adopting the intermediate energy spectrum from the work of Bethe et al., we can include higher-order corrections in our work as has been done in theirs. Besides, the short-range part of the reaction matrix G^S in our work is of Bethe et al., with the exception that here the

energy-gap parameter has not been introduced directly and the role of the hard core in determination of the energy spectrum has been reduced. In the methods of Moszkowski - Scott and Bethe, the short range reactions matrix makes a positive contribution which cancels part of the long - range effect, while in our work G^S is zero, and we get a larger separation distance than theirs, hence a weaker potential outside and a better convergence for the perturbation expansion.

The result of our calculation shows that, instead of $K_f = 1.5 F^{-1}$, the saturation point is at $K_f = 1.43 F^{-1}$, closer to the correct density $\rho = \frac{2K_f^2}{3\pi^2}$ with $K_f = 1.4 F^{-1}$. The binding energy at this density is 10 MeV/particle. (see Fig. 3). We have only calculated the first iteration for the separation distance. Since the total correction to the potential energy is small compared with the first order, we believe that this is a good approximation. The discrepancy between our calculation and observed value is still about 5 MeV/particle.

Recently much work has been done to account for this discrepancy. Bethe⁽²⁹⁾ has studied higher order diagrams in Goldstone expansion. His estimate in third order diagrams shows that they are attractive and indeed very important. His estimate is that the third order diagrams

Fig. 3. The Kinetic Energy, Potential Energy
(including first and second order corrections) and
Binding Energy per Particle as a Function of K_f^2 .



approximately cancel the correction of Brown, Schappert, and Wong⁽²⁸⁾ and bring the energy back to the value of Brueckner and Masterson. Moszkowski⁽³⁰⁾ have also found that the short range three body correlations will contribute to binding energy less than 2 MeV/particle.

The binding energy with the third order corrections is still smaller than the observed 15.5 MeV. To obtain a larger binding energy, Wong⁽²⁶⁾ has suggested to replace hard core by a soft repulsive core. For Breit's potential this contributes about 1.5 MeV.

Chapter 5.

Variational Method

Consider the total Hamiltonian of a system with A particles containing the sum of kinetic energy of the particles and two-body interactions which is given by

$$H = \sum_i T_i + \frac{1}{2} \sum_{i,j=1}^A V(\vec{r}_{ij}), \quad (5.1)$$

and let $|\psi\rangle$ be the exact wave vector and E the total energy of this system, i.e.

$$H |\psi\rangle = E |\psi\rangle \quad (5.2)$$

It can be shown by variational method⁽²⁷⁾ that for Brueckner theory the exact wave function satisfies the following equation:

$$|\psi\rangle = |\phi\rangle + \frac{Q}{e} V |\psi\rangle \quad (5.3)$$

where $|\phi\rangle$ is plane wave vector. In terms of partial wave in angular momentum space, (5.3) can be written as

$$|\psi_\ell(K)\rangle = |\phi_\ell(K)\rangle + \frac{Q}{e} V |\psi_\ell(K)\rangle \quad (5.4)$$

Equation (5.4) can be expanded in coordinate space in the following way:

$$|\psi_\ell(K)\rangle = |\phi_\ell(K)\rangle + \int d\vec{K}' d\vec{r}' d\vec{r}'' |K'\rangle \langle K'| \frac{Q}{e} |r'\rangle \langle r'| V |r''\rangle \langle r''| \psi_\ell(K)\rangle \quad (5.5)$$

or in wave function notation:

$$\begin{aligned} \psi_\ell(r) &= \phi_\ell(r) + \frac{2}{\pi} \int_0^\infty K'^2 dK' \frac{Q(K')}{e(K')} \int_0^\infty \phi_\ell(K',r) \phi_\ell(K',r') V(r') \psi_\ell(r') r'^2 dr' \\ &= \phi_\ell(r) + \int_0^\infty G'_\ell(r,r') V(r') r' \psi_\ell(r') dr'. \end{aligned} \quad (5.6)$$

Here

$$G'_\ell(r, r') = \frac{2}{\pi} \int_0^\infty \phi_\ell(K, r) \phi_\ell(K', r') K'^2 r' \frac{Q(K')}{e(K')} dK' \quad (5.7)$$

Put $U_\ell(r) = r\psi_\ell(r)$

Then (5.6) and (5.7) become

$$U_\ell(r) = rj_\ell(Kr) + \int_0^\infty G_\ell(r, r') V(r') U_\ell(r') dr' \quad (5.8)$$

and

$$G_\ell(r, r') = \frac{2}{\pi} \int_0^\infty rj_\ell(K'r) r' j_\ell(K'r') K'^2 \frac{Q(K')}{e(K')} dK' \quad (5.9)$$

For potentials with hard core (5.8) can be written as:

$$U_\ell(r) = rj_\ell(Kr) + \lambda G_\ell(r, c) + \int_c^\infty G_\ell(r, r') V(r') U_\ell(r') dr' \quad (5.10)$$

with the boundary condition $U_\ell(c) = 0$ where c is the radius of the hard core. This implies that

$$cj_\ell(Kc) + \lambda G_\ell(c, c) + \int_c^\infty G_\ell(r, r') V(r') U_\ell(r') dr' = 0 \quad (5.11)$$

By solving for λ from (5.11) and substituting into (5.10) we get

$$U_\ell(r) = f_\ell(r) + \int_c^\infty K_\ell(r, r') V(r') U_\ell(r') dr' \quad (5.12)$$

Here

$$f_\ell(r) = rj_\ell(Kr) - cj_\ell(Kc) \frac{G_\ell(c, r)}{G_\ell(c, c)} \quad (5.13)$$

and

$$K_\ell(r, r') = G_\ell(r, r') - \frac{G_\ell(r, c) G_\ell(c, r')}{G_\ell(c, c)} \quad (5.14)$$

The reaction smplitude g_ℓ is defined by

$$g_\ell = \langle f | V | U_\ell \rangle \quad (5.15)$$

To find the exact solution $|U_\ell\rangle$ we take a trial function $|U_{\ell T}\rangle$ and put

$$A[U_{\ell T}] = \langle f_\ell | V | U_{\ell T} \rangle \quad (5.16)$$

$$B[U_{\ell T}] = \langle U_{\ell T} | V - VKV | U_{\ell T} \rangle \quad (5.17)$$

$$\text{where } KV U_{\ell T}(r) = \int_0^\infty K_\ell(r, r') V(r') U_{\ell T}(r') dr' \quad (5.18)$$

Consider the functional

$$J[U_{\ell T}(r)] = \frac{A^2}{B} \quad (5.19)$$

which depends on the function $U(r)$. As an obvious consequence of (5.12) and (5.15) we have

$$J[U_\ell(r)] = g_\ell \quad (5.20)$$

Here $J[U_\ell(r)]$ depends neither on the normalization of the function U (it does not change when U is multiplied by an arbitrary constant), nor on the value taken by U in regions $V(r) = 0$. Therefore $J[U_\ell]$ takes the same value g_ℓ for any function U_ℓ satisfying the following less restrictive condition than (5.12)

$$U_\ell(r) = Dj_\ell(Kr) + KV U_\ell(r) \quad \text{if } V(r) \neq 0 \quad (5.21)$$

here D is an arbitrary constant.

Now let us calculate the variation of δJ as a function of $\delta U_{\ell T}$. From definition (5.19)

$$\delta J = \frac{2A}{B} \delta A - \frac{A^2}{B^2} \delta B \quad (5.22)$$

Since V is real and $K(r, r')$ is symmetrical in r and r' we have,

$$\delta A = \langle f_\ell | V | \delta U_{\ell T} \rangle \quad (5.23)$$

$$\delta B = 2 \langle U_{\ell T} | V - VKV | \delta U_{\ell T} \rangle \quad (5.24)$$

By substituting (5.23) and (5.24) into (5.22) we get

$$\begin{aligned} \delta J &= \frac{2A}{B^2} \{ \langle f_\ell | V | \delta U_{\ell T} \rangle \langle U_{\ell T} | V - VKV | U_{\ell T} \rangle - \langle f_\ell | V | U_{\ell T} \rangle \\ &\quad \langle U_{\ell T} | V - VKV | \delta U_{\ell T} \rangle \} \\ &= \frac{2A}{B^2} \{ \langle U_{\ell T} | V - VKV | U_{\ell T} \rangle \langle f_\ell | V | - \langle f_\ell | V | U_{\ell T} \rangle \\ &\quad \langle U_{\ell T} | (V - VKV) | \} \delta U_{\ell T} \} \\ &= \frac{2A}{B^2} \{ B \langle f_\ell | V - A \langle U_{\ell T} | (V - VKV) \rangle | \delta U_{\ell T} \rangle \\ &= \frac{2A}{B^2} \langle F_\ell | \delta U_{\ell T} \rangle \end{aligned} \quad (5.25)$$

where

$$\langle F_\ell | = B \langle f_\ell | V - A \langle U_{\ell T} | (V - VKV). \quad (5.26)$$

In order to have $\delta J = 0$ for any $|\delta U_{\ell T}\rangle$, it is necessary and sufficient that $\langle F_\ell |$ should vanish. For this it is necessary and sufficient that $\langle f_\ell |$ and $\langle U_{\ell T} | (1 - KV)$ be proportional at any point where $V(r)$ is not zero, that is, $U_{\ell T}$ be one of the functions U_ℓ obeying equation (5.21). Thus, the stationary value of J is equal to the reaction

amplitude that we wish to calculate:

$$g_{\ell} = \text{extremum of } J$$

If the potentials are also dependent on spin and isospin then the reaction matrix is related to the reaction amplitude as follows:

$$\langle K | G | K \rangle = \frac{\pi}{2} \sum_{J,T} (2J+1)(J+1) g_{\ell,J}^{S,T} \quad (5.28)$$

and the potential energy per particle contributing to the binding energy is

$$\frac{\text{P.E.}}{A} = \frac{K_f^3}{6\pi} \sum_{J,T} (2J+1)(2T+1) g_{\ell,J}^{S,T} = \frac{K_f^3}{6\pi} \sum_{J,T} (2J+1)(2T+1) J_{\ell,J}^{S,T} \Big|_{ST} \quad (5.29)$$

The function J can be written in the integral form in the following way:

$$J[U_{\ell T}] = \frac{\left[\int_c^\infty U_{\ell T}(r) V_{\ell,J}^{S,T}(r) f_{\ell}(r) dr \right]^2}{\left[\int_c^\infty U_{\ell T}(r) V_{\ell,J}^{S,T}(r) U_{\ell T}(r) dr \right] - \int_c^\infty \int_c^\infty U_{\ell T}(r) V_{\ell,J}^{S,T}(r) K_{\ell}(r,r') V_{\ell,J}^{S,T}(r') U_{\ell T}(r') dr' dr} \quad (5.30)$$

where $f_{\ell}(r)$, $G_{\ell}(r,r')$ and $K_{\ell}(r,r')$ are given in (5.9), (5.13) and (5.14).

We choose the trial function as the difference between the plane wave and a term which corresponds to the distortion,

$$U_{\ell T}(r) = rj_{\ell}(Kr) - cj_{\ell}(Kc) \frac{h_{\ell}^{(1)}(ix_{\ell}r)}{h_{\ell}^{(1)}(ix_{\ell}c)} \quad (5.31)$$

which satisfies the boundary conditions

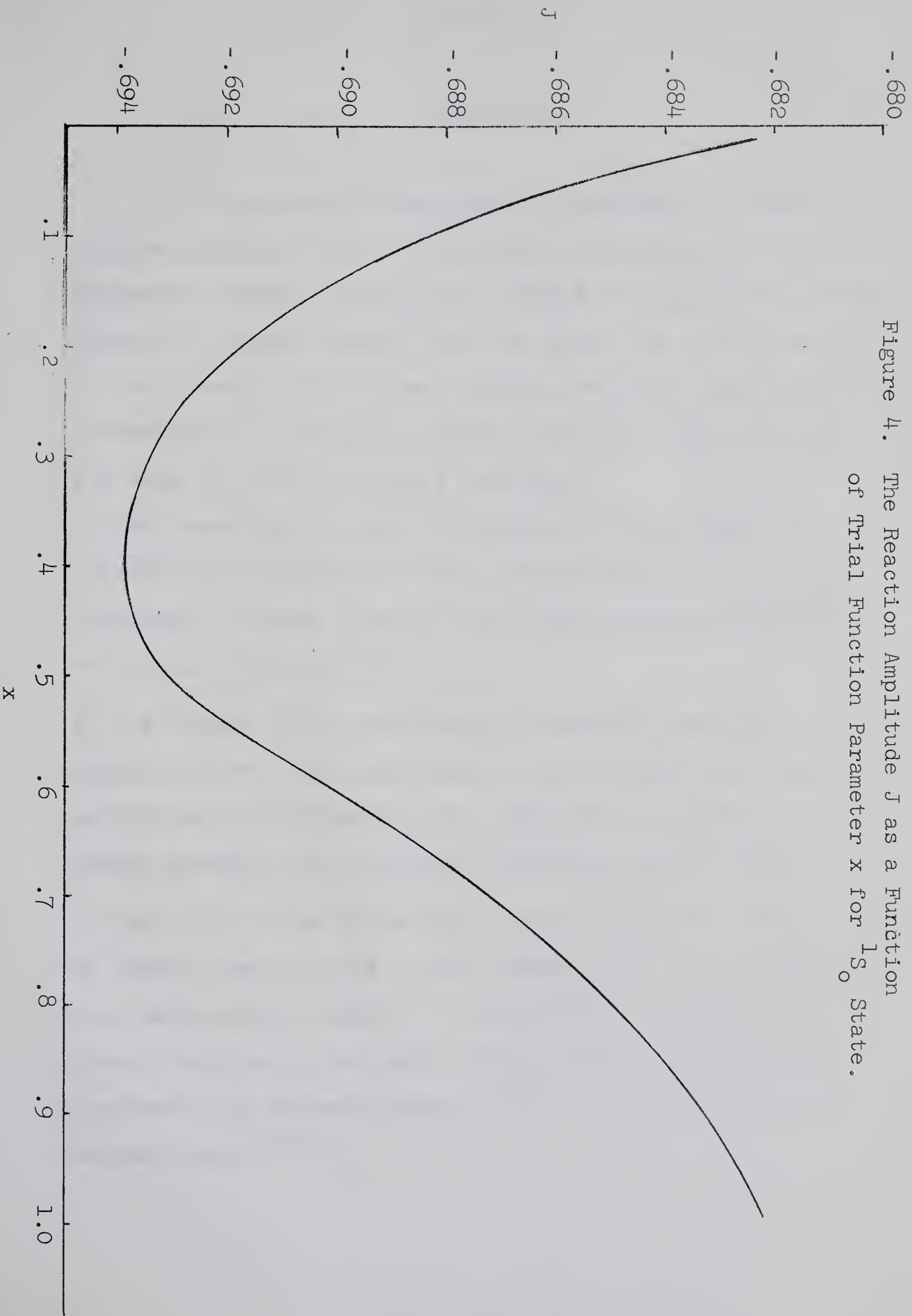
$$U_{\ell T}(c) = 0 \quad (5.32)$$

$$U_{\ell T}(\infty) = rj_{\ell}(Kr) \quad (5.33)$$

In (5.31) x_{ℓ} is the parameter which dependent on angular momentum, spin, and isospin.

In Fig. 4 the value of J as a function of the parameter x_{ℓ} for 1S_0 is plotted. The stationary value of J is found to be at the point $x = .4$. Then from (5.29) we get the contribution potential energy -15.45 MeV for 1S_0 state. The value calculated from the separation method stated in previous chapters is -14 MeV.

Figure 4. The Reaction Amplitude J as a Function of Trial Function Parameter x for 1S_0 State.



Chapter 6.

CONCLUSION

Our results seem better than others, (except for the original work of Brueckner and Gammel), but the agreement between theory and accepted value for the binding energy of nuclear matter does not imply the correctness of the theory. For nuclear matter, we still lack enough information to develop a complete theory. The following are some of the fashionable questions.

1) The uncertainty about the actual binding energy of nuclear matter because of the extrapolation of the Bethe-Weizsäcker formula (Green in the International Conference on Nuclear Physics)⁽³¹⁾.

2) The region of the validity of potential model of nuclear forces - We know that at low energies the interaction may be represented by a potential, however, at higher energies this potential model may not be valid.

3) The role of the three body forces in nuclear matter - We ignore these forces in the theory, but lots of people have calculated a specific 3 body force and the results always turn out to be large. (G.E. Brown, International Conference on Nuclear Physics⁽³¹⁾, Lecture on Theory of Nuclear Matter⁽¹⁶⁾.)

4) The convergence (or divergence) of perturbation theory - Here we have assumed a rapid convergence, but as Baker⁽¹²⁾ has shown, in general, perturbation theory diverges. In many cases, the correct result may be obtained in spite of the divergence of the series. But this cut-off process contradicts the internal consistency of the theory as in quantum electrodynamics.

5) Higher order clusters - Bethe⁽²⁹⁾ has pointed out that the three-body correlation is important and fourth order is small enough to be negligible, but the convergence (or divergence) for higher order cluster is still not clear.

Although the theory of nuclear matter is still far from being comprehensive, we agree with Bethe that it is at last at a point where "it can be used with some confidence to calculate quantities which have not been accurately observed, like compressibility, symmetry and surface energy, and where it is also applied to finite nuclei."

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